CAAM 453: Numerical Analysis I

Problem Set 1
Due: September 20th 2017

Note: All MATLAB *functions mentioned in this homework assignment can be found on the CAAM453 homepage, or come with* MATLAB*. You can use the* MATLAB *codes posted on the CAAM453 web-page. If you modify these codes, please turn in the modified code. Otherwise you do not have to turn in printouts of the codes posted on the CAAM453 web-page. Turn in all* MATLAB *code that you have written and turn in all output generated by your* MATLAB *functions/scripts.* MATLAB *functions/scripts must be commented, output must be formatted nicely, and plots must be labeled.*

Problem 1 (30 points) In laying water mains, utilities must be concerned with the possibility of freezing. Although soil and weather conditions are complicated, reasonable approximations can be made on the basis of the assumption that soil is uniform in all directions. In that case the temperature in degrees Celsius $T(x,t)$ at a distance x (in meters) below the surface, t seconds after the beginning of a cold snap, approximately satisfies

$$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right),$$

where T_s is the constant temperature during a cold period, T_i is the initial soil temperature before the cold snap,

$$\alpha = \frac{k}{\rho c_p}$$

is the thermal diffusivity (in meters^2 per second), k is the thermal conductivity (in $W/(m \cdot K)$), ρ is the density (in kg/m^3), c_p is the specific heat capacity (in $J/(kg \cdot K)$) and

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-s^2) ds$$

Assume that $T_i = 20$ [degrees C], $T_s = -15$ [degrees C], $\alpha = 0.138 \cdot 10^{-6}$ [meters^2 per second].

i. (10 points) We want to determine how deep a water main should be buried so that it will only freeze after 60 days exposure at this constant surface temperature.
Formulate the problem as a root finding problem $f(x) = 0$. What is f and what is f'? Plot the function f on $[0, \bar{x}]$, where $\bar{x}$ is chosen so that $f(\bar{x}) > 0$.

ii. (10 points) Compute an approximate depth using the Bisection Method with starting values $a_0 = 0$ [meters] and $b_0 = \bar{x}$ [meters]. You have to use the generic bisection code that has been posted on the CAAM453 web-page.

iii. (10 points) Compute an approximate depth using Newton's Method with starting value $x_0 = 0.01$ [meters]. Here, you have to write the code yourself. What happens if you start with $x_0 = \bar{x}$?

Problem 2 (25 points) Outside the region of quadratic convergence, the Newton method can be very sensitive to its choice of start point. This example will illustrate the chaotic behavior of Newton's method. Consider the complex valued function

$$f(z) = z^3 - 2z + 2$$

where $z \in C$. The equation $f(z) = 0$ has exactly 3 solutions that we call z_k, $k = 1, 2, 3$ in the complex plane. Your first goal is to find the three z_k's using any method (by hand or using a root solver).

Newton's method start with an initial guess and there are initial guesses for which the Newton iteration does not converge to any root. Your second goal will be to code your own Newton method and apply it to solve $f(z) = 0$ with initial guesses that sample a part of the complex plane.

You have to code a Newton method to apply to this particular function. The calling sequence should be

```
M   =  mynewton(n, l);
Input:
   n   is the number of starting guesses in each direction
       (real and imaginary)
   l   is the length of the real and imaginary axes used
       (the sampling domain should be centered at the origin)
Output:
   M   is a n x n matrix that contains the results
```

Line i and column j of the output matrix M correspond to a given starting guess. For a given starting guess, Newton's method will either converge to one of the z_k's or will not converge. Therefore, the output matrix M will only contain 4 distinct value (4 colours) that correspond to the 4 possible terminations of Newton's method.

Yous should be careful on detecting whether Newtons method diverge. Here, you can simply decide on a maximum number of iterations, typically 20.

Show the results as a 2D plot. You may use the `imagesc` MATLAB function to render matrix M. What you should see is called a Newton fractal.

Try to figure out where are the points for which the Newton iteration does not converge. What happens when Newtons method does not converge?

Problem 3 (15 points) Let x_0, x_1 be two successive points from a secant method applied to solving $f(x) = 0$ with $f_0 = f(x_0), f_1 = f(x_1)$. Show that regardless of which point x_0 or x_1 is regarded as the most recent point, the new point derived from the secant step will be the same.

Problem 4 (30 points) *This problem is pledged!*
Describe a problem of the form "Solve $f(x) = 0$" from your major (or discipline). Your description should have the same level of detail (including units) as Problem 1 given above. In addition to the description as a "word problem", give the relevant mathematical function $f(x)$ and also the derivative $f'(x)$. Please DO NOT discuss your choice with your classmates. I would like to see a variety of problems. If you cannot think of such a problem, I suggest you ask one of the professors in your major for an example.