MATH/CAAM 423: Partial Differential Equations

Problem Set 9: Separation of Variables and Fourier Series
Due: Friday, December 2, 2016

Problem 1: Elliptic Regularity à la Poisson (20 points)

Let $h \in L^2(\mathbb{S}^1)$, and let u be the unique solution of the Poisson equation on the unit disc $\mathbb{D} = \{x^2 + y^2 < 1\}$:

$$\begin{cases} \Delta u = 0, & x \in \mathbb{D}, \\ u\big|_{\mathbb{S}^1} = h. \end{cases}$$

Using the Poisson kernel, carefully show that u must be smooth — that is, $u \in C^\infty(\mathbb{D})$. This is an instance of *elliptic regularity*, the property that solutions of elliptic equations with smooth coefficients (but possibly non-smooth boundary conditions) are still smooth.

Problem 2: A Slice of Laplace (20 points, pledged)

Do problem 12.2 in the textbook.

Problem 3: Averaging Dirichlet Kernels for the Win (40 points)

Do problem 14.7 in the textbook. Additionally, graph the Dirichlet kernels K_N and Fejér kernels $\tilde{K}_N$ for $N = 2, 4, 8, 16$. Are your results in line with part (ii)?

Note: it will probably be useful to read the additional material for this chapter on the Dirichlet kernel before starting.

Problem 4: Jumps and Fourier Series (20 points)

Suppose $h(\theta)$ is a piecewise constant function on $\mathbb{S}^1$. Now, apply the Poisson kernel to $h(\theta)$:

$$u(r,\theta) = \int_{-\pi}^{\pi} K_P(r, \theta - \omega) h(\omega)\, d\omega.$$

Find the pointwise limit $\lim_{r \to 1} u(r,\theta)$. In particular, what happens at h's points of discontinuity?