MATH/CAAM 423: Partial Differential Equations

Problem Set 5: Fourier Transform
Due: Thursday, October 20, 2016

Problem 1: Fourier Transform Computations (20 points)
Do problems 8.1 and 8.3 in the textbook.

Problem 2: The Schrödinger Equation (30 points)
Do problem 8.9 in the textbook.

Problem 3: The Radon Transform (20 points)
Do problem 8.11 in the textbook.

Problem 4: Addendum on the Radon Transform (10 points)
The Radon transform R was defined in the last problem we looked at (8.11 in the textbook). Suppose u is a solution of Poisson's equation:

$$\Delta u = f,$$

for some compactly supported function f. Find a (non-trivial) PDE that Ru must satisfy. What can you say about the Radon transform of a harmonic function?

Problem 5: Heat Diffusion (20 points, pledged)
From everyday experience, we know that heat diffuses: heat concentrated in a small area on an otherwise cold object will eventually spread through the whole object. As well, any sudden changes (or discontinuities) in temperature will quickly spread out. We can analyze this behavior nicely with the Fourier transform.

Let $f \in L^1(\mathbb{R}^n)$, and let $u(t,x)$ be the solution of the heat equation with initial value f:

$$\begin{cases} u_t = \Delta u, \\ u\big|_{t=0} = f. \end{cases}$$

(a) Prove that for any starting heat distribution $f \in L^1$, the solution u at a fixed later time $T > 0$ is a smooth (C^∞) function in space. (So, as a result, any sudden jumps in f are instantly smoothed out after $t = 0$.)

(b) Find $\lim_{t\to\infty} u(t,x)$.