MATH/CAAM 423: Partial Differential Equations

Problem Set 4: Maximum Principle
Due: Friday, October 7, 2016

Problem 1: Maximum Principle for Heat Equation (20 points)

Do problem 7.7 in the textbook.

Problem 2: Elliptic Energy Estimates (20 points)

Do problem 7.8 in the textbook.

Problem 3: Means of Harmonic Functions (15 points)

A solution of Laplace's equation, $\Delta u = 0$, is often called a *harmonic* function. Harmonic functions have interesting properties. In this problem you will prove one of them, the mean value property.

We will work in $\mathbb{R}^3$ in this problem but the result holds for all dimensions. Write $S_r(x)$ for the sphere of radius r centered at x.

The mean value property says that if u is continuous and $\Delta u = 0$ inside $S_r(x)$ for some $x \in \mathbb{R}^3$ and $r > 0$, then

$$u(x) = \frac{1}{\operatorname{vol} S_r(x)} \int_{S_r(x)} u(z)\, dS(z). \tag{1}$$

Here $\operatorname{vol} S_r(x)$ is the surface area of $S_r(x)$. In other words, $u(x)$ is the average of u's values on a sphere around x. Prove this.

Hint: Divergence theorem!

Problem 4: Strong Maximum Principle (15 points)

In class, we proved what is sometimes known as the *weak* maximum principle for Laplace's equation: if $\Delta u = 0$ on a bounded connected domain $\Omega \subset \mathbb{R}^n$, then the maximum of u is obtained at the boundary. We may state it this way:

$$\sup_{x \in \Omega} u \leq \max_{\partial\Omega} u.$$

Of course, u can achieve its maximum at multiple points, and all the statement above says is that at least one of these maxima is on the boundary.

However, something stronger is true: *all* the maxima are on the boundary, unless u is constant! In other words, a nonconstant u cannot have a maximum in the interior. Use Problem 3 to prove this.

You may assume $n = 3$ (although as remarked in Problem 3, it is not necessary to do so).

Problem 5: Energy Conservation for Acoustic and Elastic Waves (30 points, pledged)

(a) Consider the variable-speed acoustic wave equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with a zero-order term:

$$u_{tt} - c^2(x)\Delta u + qu = 0,$$

where $c(x) \in C^0(\mathbb{R}^n)$, with either Dirichlet or Neumann boundary conditions:

$$u\Big|_{\partial\Omega} = a \qquad\qquad or \qquad\qquad \frac{du}{d\nu}\Big|_{\partial\Omega} = 0,$$

for constant a. Find an energy functional E (independent of u) that is always conserved; that is, for any solution $u(t,x)$ the function $E(u(t,\cdot))$ is constant in time.

(b) Now, we turn to the elastic wave equation. Here $\mathbf{u} = (u^{(1)}, \dots, u^{(n)})$ is a vector-valued function on a bounded subset $\Omega \subset \mathbb{R}^n$. (Recall it represents the displacement of the body at each point). The elastic wave equation with constant parameters λ, μ is

$$\mathbf{u}_{tt} = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\operatorname{div} u).$$

We assume Dirichlet boundary conditions: $\mathbf{u}|_{\partial\Omega} = 0$. (In Euclidean space, the vector Laplacian $\Delta\mathbf{u}$ is just the Laplacian of each component: $\Delta\mathbf{u} = (\Delta u^{(1)}, \dots, \Delta u^{(n)})$).

Find an energy functional E (independent of $\mathbf{u}$) that is conserved by all solutions $\mathbf{u}$. Again, this means that $E(\mathbf{u}(t,\cdot))$ is constant in t if $\mathbf{u}$ is a solution of the elastic wave equation.