MATH/CAAM 423: Partial Differential Equations

Problem Set 2: Distributions
Due: September 21, 2016

Problem 1: Wave Travel (20 points)

The equation

$$u_t + cu_x = 0$$

is an *advection equation*, and it describes waves travelling in one direction in 1D. The constant $c \neq 0$ here is the wave speed. Find a distribution solving this equation that is not equal to an ordinary function. Make sure to prove that it really is a solution!

Problem 2: Weak Solutions to Burgers' Equation (30 points)

Do problem 5.7 and the first part of 5.8 from the textbook (omitting the entropy condition part of 5.8).

Problem 3: Uniqueness of Derivatives (20 points)

(a) Suppose $u, v \in \mathcal{D}'(\mathbb{R})$ are two distributions with $u' = v'$. Prove that $u = v + c$ for some constant c.

(b) Suppose $u' = f$, where f is a continuous, compactly supported function on $\mathbb{R}$. Using (a), show that u is a C^1 function.

Problem 4: A Restriction Problem (30 points, pledged)

The restriction $u|_X$ of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ to a subset $X \subseteq \mathbb{R}^n$ is the restriction of u to C_∞ functions whose support is inside X.

Therefore, $u|_X = f$ (where f is a continuous function on X) means that

$$u(\phi) = \int_X f(x)\phi(x)\,dx \qquad \text{for all } \phi \in C^\infty(\mathbb{R}^n) \text{ s.t. } \operatorname{supp}\phi \subset X.$$

If u is actually a function itself, the restriction $u|_X$ is the same as the usual restriction of a function to a subset of its domain.

(a) If f, g are any two bounded, continuous functions on $\mathbb{R}$, prove that there is a distribution u such that $u\big|_{x>0} = f$ and $u\big|_{x<0} = g$. *Be careful! Your distribution must be defined on* all *test functions* $\phi \in C_c^\infty(\mathbb{R})$.

(b) Find a one-dimensional distribution $u \in \mathcal{D}'(\mathbb{R})$ such that $u\big|_{x>0} = 1/x$ and $u\big|_{x<0} = 0$.