

## Chapter 3

# Microlocalization

### 3.1 Singular support and wavefront set

The core objective of microlocal analysis is to study how the singularities of a distribution change under certain types of mappings. Perhaps the original application of microlocal analysis, and the one in which we are mostly interested, is the study of how the singularities of a distribution move under the solution operator of a differential equation. In order to begin studying this subject we first define what we mean by the singularities of a distribution. There are two different, related, ways in which we characterize these singularities. The first is the singular support.

**Definition 3.1 (Singular support).** *Suppose that  $A \in \mathcal{D}'(X)$  where  $X \subset \mathbb{R}^n$  is open. Then the singular support of  $A$  is defined by*

$$\text{singsupp}(A) = \left\{ \bigcup U : U \subset \mathbb{R}^n \text{ is open, and } A|_U = T_f \text{ for some } f \in C^\infty(U) \right\}^c.$$

*Intuitively the singular support is the complement of the largest set in which  $A$  is represented by a smooth function.*

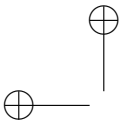
The second is the wavefront set, which considers not only the locations where  $A \in \mathcal{D}'(\mathbb{R}^n)$  is singular but also the directions.

**Definition 3.2 (Wavefront set).** *Suppose that  $A \in \mathcal{D}'(X)$  where  $X \subset \mathbb{R}^n$  is open. Then  $(x_0, \eta_0) \in X \times (\mathbb{R}^n \setminus \{0\})$  is in the complement of the wavefront set of  $A$  (which is denoted  $WF(A)$ ) if and only if there exists a neighborhood  $U$  of  $x_0$  and  $V$  of  $\eta_0$  such that for every  $\varphi \in C_c^\infty(U)$  and  $N \in \mathbb{R}$  there exists a constant  $C$  such*

$$|\mathcal{F}[\varphi A](\lambda\eta)| \leq C\lambda^{-N}$$

*for all  $\eta \in V$ .*

Study of the wavefront set of distributions is the core of microlocal analysis. Indeed, we will



say that two distributions  $A$  and  $A' \in \mathcal{D}(X)$  are equivalent microlocally if  $WF(A - A') = \emptyset$  which, by theorem 3.3 below, implies that  $A - A' \in C^\infty(X)$ .

There are a few very basic facts about the wavefront set that we should mention now before proceeding. First, it is clear that  $WF(A)$  is a conic set, which means that if  $(x, \eta) \in WF(A)$  then also  $(x, \lambda\eta) \in WF(A)$  for all  $\lambda \in \mathbb{R}^+$ . Second, by definition, the complement of  $WF(A)$  is open from the definition, and so  $WF(A)$  is a closed subset of  $(X \times (\mathbb{R}^n \setminus \{0\}))$ . Finally we mention for readers familiar with differential geometry that the wavefront set can be thought of as a subset of the cotangent bundle  $T^*X$ . This intuitively matches with the notion that elements of  $WF(A)$  are the normal vectors to submanifolds of  $X$  where  $A$  has singularities.

The singular support and wavefront set of a distribution are related by the following theorem.

**Theorem 3.3.** *Suppose that  $A \in \mathcal{D}'(X)$  where  $X \subset \mathbb{R}^n$  is open. Then*

$$\text{singsupp}(A) = \pi_x(WF(A))$$

where  $\pi_x : X \times (\mathbb{R}^n \setminus \{0\}) \rightarrow X$  is the projection onto  $X$ .

**Proof.** We prove instead that

$$\text{singsupp}(A)^c = \pi_x(WF(A))^c.$$

Suppose that  $x_0 \in \text{singsupp}(A)^c$ . Then by definition there exists a neighborhood  $U$  of  $x_0$  such that when restricted to  $U$ ,  $A$  is represented by a smooth function. Let us then take a function  $\varphi \in C_c^\infty(U)$  such that  $\varphi(x_0) = 1$ . Then  $\varphi A \in C_c^\infty(\mathbb{R}^n)$  and therefore  $\mathcal{F}[\varphi A] \in \mathcal{S}(\mathbb{R}^n)$ . Thus for any  $\eta_0 \in \mathbb{R}^n \setminus \{0\}$ ,  $\mathcal{F}[\varphi A](\lambda\eta_0)$  decreases faster than any power of  $\lambda$  as  $\lambda \rightarrow \infty$ . Therefore  $x_0 \in \pi_x(WF(A))^c$  and so

$$\text{singsupp}(A)^c \subset \pi_x(WF(A))^c.$$

Now we work to prove the opposite inclusion.

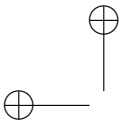
Suppose that  $x_0 \in \pi_x(WF(A))^c$ . Then for every  $\eta_0 \in \mathbb{R}^n \setminus \{0\}$ ,  $(x_0, \eta_0) \in WF(A)^c$ . For every such  $\eta_0 \in \mathbb{S}^{n-1}$  let  $V_{\eta_0}$  be the neighborhood which we know exists by the definition of the wavefront set. Since  $\mathbb{S}^{n-1}$  is compact, it can be covered by a finite number of such  $V_{\eta_0}$  corresponding to say  $\{\eta_j\}_{j=1}^k$ . By taking the intersection of the corresponding neighborhoods  $U_{\eta_j}$  of  $x_0$  and the largest constants we see that in fact there exists a neighborhood  $U$  of  $x_0$  such that for any  $N \in \mathbb{R}$  there is a constant  $C > 0$  such that for any  $\varphi \in C_c^\infty(U)$  and  $\eta \in \mathbb{S}^{n-1}$

$$(3.1) \quad |\mathcal{F}[\varphi A](\lambda\eta)| \leq C\lambda^{-N}.$$

Let us now show that this implies  $\varphi A \in C_c^\infty(X)$ . Indeed, by the Fourier inversion formula

$$\varphi A(x) = \frac{1}{(2\pi)^n} \int e^{ix \cdot \xi} \mathcal{F}[\varphi A](\xi) d\xi.$$

From (3.1) we see that the integrand above is bounded by an integrable function for all  $x$ , and so by the dominated convergence theorem  $\varphi A$  is a continuous function. Further, we



can take the derivative under the integral with respect to  $x$  and this remains true, and so  $\varphi A \in C_c^\infty(U)$ . Finally, suppose that  $U' \Subset U$  is another neighborhood of  $x_0$ , and take a  $\varphi \in C_c^\infty(U)$  such that  $\varphi|_{U'} = 1$ . Then

$$\varphi A|_{U'} = A \in C^\infty(U')$$

and so  $x_0 \in \text{singsupp}(A)^c$ .  $\square$

To finish this section we will show another way to characterize the wavefront set of a distribution which will be useful later. To motivate this first consider that for  $A \in \mathcal{D}'(X)$

$$\mathcal{F}[\varphi A](\lambda\eta) = A(e^{-i\lambda\eta \cdot x} \varphi(x)).$$

As it turns out, it may be useful at times to replace the function  $\eta \cdot x$  by something else. This is possible as shown in the next lemma.

**Lemma 3.4.** *If  $A \in \mathcal{D}'(X)$ , then  $(x_0, \eta_0) \in WF(A)^c$  if and only if for every  $\psi(x, \eta) \in C^\infty(X \times \mathbb{R}^n)$  such that  $\partial_x \psi(x_0, 0) = \eta_0$  there exists a neighborhood  $U \subset X$  of  $x_0$  and a neighborhood  $V \subset \mathbb{R}^n$  of 0 such that for all  $\varphi \in C_c^\infty(U)$  and  $N > 0$  there exists a constant  $C$  such that*

$$|A(e^{-i\lambda\psi(x, \eta)} \varphi(x))| \leq C\lambda^{-N}$$

for all  $\eta \in V$ .

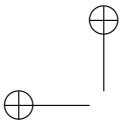
**Proof.** The “if” half of the statement is immediate if we take  $\psi(x, \eta) = x \cdot (\eta_0 - \eta)$ . For the other half assume that  $(x_0, \eta_0) \in WF(A)^c$ , and suppose  $U$  is the neighborhood of  $x_0$  and  $V$  the neighborhood of  $\xi$  which exist from the definition of the wavefront set. Let  $\varphi \in C_c^\infty(U)$  and let  $\tilde{\varphi} \in C_c^\infty(U)$  be equal to 1 on  $\text{supp}(\varphi)$ . Then

$$\begin{aligned} A(e^{-i\lambda\psi(x, \eta)} \varphi(x)) &= \int \mathcal{F}^{-1}[e^{-i\lambda\psi(x, \eta)} \tilde{\varphi}(x)](\xi) \mathcal{F}[\varphi A](\xi) \, d\xi \\ &= \int \left( \int e^{i(x \cdot \xi - \lambda\psi(x, \eta))} \tilde{\varphi}(x) \, dx \right) \mathcal{F}[\varphi A](\xi) \, d\xi \\ &= \frac{\lambda^N}{(2\pi)^n} \int \left( \int e^{i\lambda(x \cdot \xi - \psi(x, \eta))} \tilde{\varphi}(x) \, dx \right) \mathcal{F}[\varphi A](\lambda\xi) \, d\xi. \end{aligned}$$

Now we let us consider the integral with respect to  $x$  in parentheses above. Provided we take  $U$  small enough, then, since  $\partial_x \psi(x_0, 0) = \eta_0$ , if we take a small enough neighborhood  $\tilde{V}$  of  $\eta_0$  and  $V$  of 0 the function  $x \mapsto x \cdot \xi - \varphi(x, \eta)$  will have no stationary points for  $x \in U$ ,  $\xi \in \tilde{V}^c$  and  $\eta \in V$ . Let us also take  $\tilde{V}$  small enough so that it satisfies the definition of the complement of the wavefront set. Now we can apply the method from the proof of lemma 2.2 to show that for any  $N \in \mathbb{R}$  there is a constant  $C$  such that

$$\left| \int e^{i\lambda(x \cdot \xi - \psi(x, \eta))} \tilde{\varphi}(x) \, dx \right| \leq C\lambda^{-N} (1 + |\xi|)^{-N}.$$

In the rest of the proof the constant  $C$  may change from step to step, but we do not reflect this in the notation. Since  $A$  is a distribution there exist constants  $C$  and  $k$  such that (for  $\lambda$



sufficiently large)

$$|\mathcal{F}[\varphi A](\lambda\xi)| = |A(e^{-i\lambda\xi \cdot x} \varphi(x))| \leq C(1 + |\lambda\xi|)^k \leq C\lambda^k(1 + |\xi|)^k.$$

Combining these estimates shows that

$$\left| \frac{\lambda^N}{(2\pi)^n} \int_{V^c} \left( \int e^{i\lambda(x \cdot \xi - \psi(x, \eta))} \tilde{\varphi}(x) \, dx \right) \mathcal{F}[\varphi A](\lambda\xi) \, d\xi \right| \leq C\lambda^{-N+k}$$

for any  $N$ . On the other hand, since  $(x_0, \eta_0) \notin WF(A)$ , for any  $N \in \mathbb{R}$  there is a constant  $C > 0$  such that

$$|\mathcal{F}[\varphi A](\lambda\xi)| \leq C\lambda^{-N}$$

for all  $\xi \in \tilde{V}$ , and therefore

$$\left| \frac{\lambda^N}{(2\pi)^n} \int_V \left( \int e^{i\lambda(x \cdot \xi - \psi(x, \eta))} \tilde{\varphi}(x) \, dx \right) \mathcal{F}[\varphi A](\lambda\xi) \, d\xi \right| \leq C\lambda^{-N}.$$

Combining these estimates completes the proof.  $\square$

It is worth noting that lemma 3.4 gives a coordinate independent way of defining the wavefront set which also naturally shows, as remarked above, that the wavefront set lies in the cotangent bundle  $T^*X$ .

We finally introduce a definition that will be necessary later.

**Definition 3.5.** *Suppose that  $\Lambda \subset X \times (\mathbb{R}^n \setminus \{0\})$  is an open conic set. Then the set  $\mathcal{D}'_\Lambda(X) \subset \mathcal{D}'(X)$  is the set of distributions whose wavefront set is contained in  $\Lambda$ . Similarly  $\mathcal{E}'_\Lambda(X)$  is the subset of  $\mathcal{E}'(X)$  with wavefront set contained in  $\Lambda$ .*

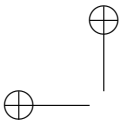
## 3.2 Wavefront set of distributions defined by oscillatory integrals

Our goal for the rest of this chapter is to investigate the wavefront set, and other smoothness properties, of distributions defined by oscillatory integrals. In order to complete this project we will have to introduce some extra hypotheses for the phase function  $\phi$ , but let us put that aside for the moment. Suppose that  $a \in S_{\rho, \delta}^\mu(X \times \mathbb{R}^N)$  with  $\delta < 1$ ,  $\rho > 0$  (more restrictions will be placed on  $\rho$  and  $\delta$  below),  $\phi \in C^\infty(X \times \mathbb{R}^N)$  is a real-valued phase function, and let  $A$  be the corresponding oscillatory integral. Let us attempt to show that a point  $(x_0, \eta_0) \notin WF(A)$  and simply see what happens. To do this let  $U$  be some small neighborhood of  $x_0$ , take some  $\varphi \in C_c^\infty(U)$  and a function  $\psi(x, \eta) \in C^\infty(X \times \mathbb{R}^N)$  such that  $\partial_x \phi(x_0, 0) = \eta_0$  and, using a representation in the form (1.15) for the oscillatory integral  $A$ , study for  $\eta$  in a neighborhood of 0 the integral

$$A(e^{-i\lambda\psi(x, \eta)} \varphi(x)) = \int \int e^{i(\phi(x, \xi) - \lambda\psi(x, \eta))} e^{i\lambda\psi(x, \eta)} (L^t)^k e^{-i\lambda\psi(x, \eta)} a(x, \xi) \varphi(x) \, d\xi \, dx.$$

Now, note that (c.f. (??))

$$\begin{aligned} L_\eta^t &:= e^{i\lambda\psi(x, \eta)} (L^t)^k e^{-i\lambda\psi(x, \eta)} \\ &= A(x, \xi) + \sum_j B_j(x, \xi) (-i\lambda \partial_x \psi(x, \eta) + \partial_{x^j}) + \sum_j C_j(x, \xi) \partial_{\xi^j} \end{aligned}$$



and  $e^{ix \cdot \eta} (L^t)^k e^{-i\lambda x \cdot \eta} = (L_\eta^t)^k$ . From this we may conclude that

$$e^{i\psi(x,\eta)} (L^t)^k e^{-i\lambda\psi(x,\eta)} a(x, \xi) \varphi(x) = \sum_{j=0}^k \lambda^j f_j(x, \xi; \eta)$$

where each  $f_j(\cdot, \cdot; \eta) \in S_{\rho, \delta}^{\mu - (k-j) \min(1-\delta, \rho) - j}(X \times \mathbb{R}^N)$  has support that when projected onto  $X$  is contained in  $\text{supp}(\varphi)$  and with constants in the estimates uniform for  $\eta$  in a neighborhood of 0. Now we can change variables in the integral above to find

$$(3.2) \quad A(e^{-i\lambda\psi(x,\eta)} \varphi(x)) = \int \int e^{i\lambda(\phi(x,\xi) - \psi(x,\eta))} \lambda^N \sum_{j=0}^k \lambda^j f_j(x, \lambda\xi; \eta) d\xi dx.$$

We can analyze this integral using the method of stationary phase. First we consider the case that the function  $\phi(x, \xi) - \psi(x, \eta)$  has no stationary points on the support of  $a(x, \xi) \varphi(x)$ . We would like to apply lemma 2.2, but unfortunately the functions  $f_j(x, \lambda\xi)$  do not fully satisfy the required hypotheses for  $a_\lambda$ . However, for any given  $M > 0$ , if we take  $k$  sufficiently large then for  $|\alpha| \leq M$

$$\|\partial_x^\alpha \partial_\xi^\beta [\lambda^{N+j} f_j(x, \lambda\xi)]\|_{L^1(X \times \mathbb{R}^N)} \leq C \lambda^{N+\mu+|\beta|(1-\rho)+|\alpha|\delta}$$

Provided  $\rho > 0$  and  $\delta < 1$  this is enough for the proof of lemma 2.2, and we see that in this case the integral decays faster than any power of  $\lambda$ .

Now let us examine in more detail when  $\phi(x, \xi) - \psi(x, \eta)$  has a stationary point. Since  $\partial_x \psi(x_0, 0) = \eta_0$ , given a neighborhood  $V$  of  $\eta_0$  we can find  $U$  sufficiently small and a neighborhood  $\tilde{V}$  of 0 such that  $\partial_x \psi(x_0, 0) \in V$  for all  $x \in U$  and  $\eta \in \tilde{V}$ . Thus, unless there exists  $\xi_0$  such that

$$(3.3) \quad \begin{aligned} \partial_\xi \phi(x_0, \xi_0) &= 0, \\ \partial_x \phi(x_0, \xi_0) &= \eta_0, \end{aligned}$$

and  $(x_0, \lambda\xi_0) \in \text{supp}(a\varphi)$  for some  $\lambda$  we can always make  $U$  and  $\tilde{V}$  sufficiently small so that there is no stationary point of  $\phi(x, \xi) - \psi(x, \eta)$  on  $\text{supp}(a\varphi)$  with  $\eta \in \tilde{V}$ . In fact we can see from the above analysis that we only need require that there is no solution of (3.3) for  $(x_0, \xi_0) \in \text{ess supp}(a\varphi)$  where the essential support of  $a$  is a closed conic subset of  $X \times \mathbb{R}^n \setminus \{0\}$  defined by

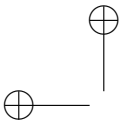
$$\text{ess supp}(a) = \left\{ \bigcup U \times \Gamma : a \in S_{\rho, \delta}^{-\infty}(U \times \Gamma) \right\}^c.$$

Summarizing what we have found above we have the following lemma.

**Lemma 3.6.**

$$WF(A) \subset \{(x, \partial_x \phi(x, \xi)) : \partial_\xi \phi(x, \xi) = 0, (x, \xi) \in \text{ess supp}(a)\}$$

In the next subsection we will introduce some extra hypotheses that imply the inclusion in lemma 3.6 is actually an equality.



### 3.3 Nondegenerate and clean phase functions

Motivated by lemma 3.6, we now begin introducing some notations and definitions. To start, we define

$$C_\phi = \{(x, \xi) \in X \times (\mathbb{R}^N \setminus \{0\}) : \partial_\xi \phi(x, \xi) = 0\}$$

and consider the map  $T_\phi$  defined on  $C_\phi$  by

$$C_\phi \ni (x, \xi) \mapsto T_\phi(x, \xi) = (x, \partial_x \phi(x, \xi)).$$

We denote the image of this map by  $\Lambda_\phi$ . Lemma 3.6 can be restated as

$$WF(A) \subset T_\phi(C_\phi \cap \text{ess supp}(a)).$$

Once we added some extra hypotheses for  $\phi$  we can establish that in fact this is an equality. These hypotheses may be motivated from two points of view. First, they guarantee that when the phase function appearing in (3.2) has a stationary point it is nondegenerate. Second they guarantee that  $C_\phi$  is a smooth manifold, and the map  $T_\phi$  is constant rank and so  $\Lambda_\phi \subset X \times (\mathbb{R}^n \setminus \{0\})$  is an immersed submanifold. In fact,  $\Lambda_\phi$  will be a conic Lagrangian submanifold, and we can take advantage of this extra structure to great utility. Now let us move on to the necessary definitions.

**Definition 3.7 (Clean phase function).** *A real-valued phase function  $\phi$  is called a clean phase function if  $C_\phi$  is a smooth manifold.*

When  $\phi$  is a clean phase function and  $(x, \xi) \in C_\phi$ , the tangent space of  $C_\phi$  at  $(x, \xi)$  is given as a subspace of  $T_{(x, \xi)}(X \times (\mathbb{R}^N \setminus \{0\}))$  by

$$T_{x, \xi} C_\phi = \left\{ a^j \partial_{x^j} + b^j \partial_{\xi^j} : a^j \partial_{\xi^l x^j}^2 \phi(x, \xi) + b^j \partial_{\xi^l \xi^j}^2 \phi(x, \xi) = 0 \right\}.$$

Since  $C_\phi$  is a smooth manifold we can see from this formula that the  $N \times (n + N)$  matrix

$$(3.4) \quad \begin{pmatrix} \partial_{\xi^k}^2 \phi(x, \xi) & \partial_{\xi^k \xi^j}^2 \phi(x, \xi) \end{pmatrix}$$

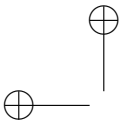
must have a constant rank equal to  $N + n - \dim(C_\phi)$  for  $(x, \xi) \in C_\phi$ . On the other hand, the differential of the map  $T_\phi$  acting on the whole space  $T_{(x, \xi)}(X \times (\mathbb{R}^N \setminus \{0\}))$  is given by (taking coordinates  $x$  and  $\eta$  on  $X \times (\mathbb{R}^n \setminus \{0\})$ )

$$(3.5) \quad a^j \partial_{x^j} + b^j \partial_{\xi^j} \mapsto a^j \partial_{x^j} + \left( a^j \partial_{x^k x^j}^2 \phi(x, \xi) + b^j \partial_{x^k \xi^j}^2 \phi(x, \xi) \right) \partial_{\eta^k}.$$

To calculate the rank of the differential  $DT_\phi$  of  $T_\phi$  acting on  $T_{(x, \xi)} C_\phi$  we calculate the intersection of  $\ker(DT_\phi)$  with  $T_{(x, \xi)} C_\phi$ . From the above formulas we find that this intersection is given by

$$\left\{ b^j \partial_{\xi^j} : b^j \partial_{x^k \xi^j}^2 \phi(x, \xi) = 0, \text{ and } b^j \partial_{\xi^l \xi^j}^2 \phi(x, \xi) = 0 \right\}.$$

Therefore  $\text{rank}(DT_\phi|_{T_{(x, \xi)} C_\phi}) = n$  and so  $\Lambda_\phi$  is an immersed submanifold of dimension  $n$ , and the map  $T_\phi : C_\phi \rightarrow \Lambda_\phi$  is a fibration with fibers of dimension  $e = \dim(C_\phi) - n$ .



The extreme case when  $e = 0$  occurs when the rank of  $\begin{pmatrix} \partial_{\xi x}^2 \phi(x, \xi) & \partial_{\xi \xi}^2 \phi(x, \xi) \end{pmatrix}$  is equal to  $N$ . In this case we call the phase function nondegenerate.

**Definition 3.8 (Nondegenerate phase function).** *If  $\phi$  is a real-valued phase function and the matrix*

$$\begin{pmatrix} \partial_{\xi x}^2 \phi(x, \xi) & \partial_{\xi \xi}^2 \phi(x, \xi) \end{pmatrix}$$

*has constant rank equal to  $N$  for  $(x, \xi) \in C_\phi$  (which is the maximum possible), then  $\phi$  is called a nondegenerate phase function.*

From the considerations above the definition we know that when  $\phi$  is a nondegenerate phase function  $C_\phi$  is a smooth manifold, the map  $T_\phi : C_\phi \rightarrow X \times (\mathbb{R}^n \setminus \{0\})$  is an immersion, and  $\Lambda_\phi$  is an immersed submanifold of  $X \times (\mathbb{R}^n \setminus \{0\})$ . We will usually assume that we have nondegenerate phase functions, but when we eventually study the clean composition calculus for Fourier integral operators we will need to consider clean but not nondegenerate phase functions.

Now let us return to the problem of calculating the wavefront set of an oscillatory integral. Suppose that  $\phi$  is a nondegenerate phase function and consider (3.2). We now assume that  $(x_0, \eta_0) \in \Lambda_\phi$ . Then there is  $\xi_0$  such that  $(x_0, \xi_0) \in C_\phi$  and  $(x_0, \xi_0)$  is a stationary point for the phase function  $\phi(x, \xi) - \psi(x, \eta)$ . In order to apply the method of stationary phase we need to check if  $(x_0, \xi_0)$  is a nondegenerate stationary point. This is the case if

$$Q = \begin{pmatrix} \partial_{xx}^2 \phi(x_0, \xi_0) - \partial_{xx}^2 \psi(x_0, 0) & \partial_{x\xi}^2 \phi(x_0, \xi_0) \\ \partial_{\xi x}^2 \phi(x_0, \xi_0) & \partial_{\xi\xi}^2 \phi(x_0, \xi_0) \end{pmatrix}$$

is invertible. That  $\phi$  is a nondegenerate phase function implies that the bottom  $N$  rows of  $Q$  are linearly independent. By choosing for example

$$(3.6) \quad \psi(x, \eta) = x \cdot \eta_0 + (x - x_0)^T \left( \frac{1}{2} \partial_{xx}^2 \phi(x_0, \xi_0) + \mathbf{I} \right) (x - x_0)$$

we can also ensure that the top  $n$  rows are linearly independent and so  $Q$  is invertible. We also mention a geometric interpretation for the invertibility of the matrix  $Q$ . Note that if

$$Q \begin{pmatrix} a \\ b \end{pmatrix} = 0,$$

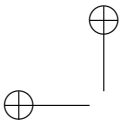
then  $a^j \partial_{x^j} + b^j \partial_{\xi^j} \in T_{(x_0, \xi_0)} C_\phi$  and so

$$a^j \partial_{x^j} + \left( a^j \partial_{x^k x^j}^2 \phi(x_0, \xi_0) + b^j \partial_{x^k \xi^j}^2 \phi(x_0, \xi_0) \right) \partial_{\eta^k} \in T_{(x_0, \eta_0)} \Lambda_\phi.$$

On the other hand, if we let  $\Lambda = \{(x, \partial_x \psi(x, 0))\}$  be the graph of the differential of  $\psi$ , then

$$T_{(x_0, \eta_0)} \Lambda = \tilde{a}^j \partial_{x^j} + \tilde{a}^j \partial_{x^k x^j}^2 \psi(x_0, 0) \partial_{\eta^j}.$$

Thus the requirement that  $Q$  be invertible is equivalent to the requirement that  $T_{(x_0, \eta_0)} \Lambda_\phi$  and  $T_{(x_0, \eta_0)} \Lambda$  be transverse.



Finally, returning to (3.2) and using the discussion in the previous paragraph and theorem 2.5, we may conclude that when  $\phi$  is a nondegenerate phase function such that  $T_\phi$  restricted to  $C_\phi \cap \text{ess supp}(a)$  is injective and  $\psi$  is chosen appropriately (as for example in (3.6)), then for  $\eta$  in a neighborhood of 0

$$(3.7) \quad \begin{aligned} & A(e^{-i\lambda\psi(x,\eta)}\varphi(x)) \\ &= \left( (2\pi)^{(n+N)/2} \lambda^{N/2-n/2} \frac{e^{i(\text{sgn}(Q)\frac{\pi}{4} + \lambda(\phi(x,\xi) - \psi(x,\eta)))}}{\sqrt{|Q|}} a(x, \lambda\xi)\varphi(x) \right) \Big|_{x=x(\eta), \xi=\xi(\eta)} \\ & \quad + O(\lambda^{N/2-n/2+\mu-\min(\rho, 1-\delta)}) \end{aligned}$$

where  $x(\eta)$  and  $\xi(\eta)$  are the unique stationary points for  $(x, \xi) \mapsto \phi(x, \xi) - \phi(x, \eta)$ . Therefore

$$|A(e^{-i\lambda\psi(x,\eta)}\varphi(x))| = O(\lambda^{N/2-n/2+\mu}), \quad |A(e^{-i\lambda\psi(x,\eta)}\varphi(x))| \neq O(\lambda^{N/2-n/2+\mu-1})$$

and so lemma 3.4 gives the following theorem.

**Theorem 3.9.** *If  $\phi$  is a nondegenerate phase function and  $T_\phi$  restricted to  $C_\phi \cap \text{ess supp}(a)$  is injective then*

$$WF(A) = T_\phi(C_\phi \cap \text{ess supp}(a)).$$

Let us look at some applications of this theorem.

**Example 3.10** Suppose that  $M$  is a submanifold of  $\mathbb{R}^n$  that is given by  $M = f^{-1}(\{0\})$  for some smooth mapping  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $|\partial_x f(x)| = 1$  on  $M$ . Then integration over  $M$  with respect to Hausdorff measure is a distribution given by the oscillatory integral

$$A = \int_{\mathbb{R}} e^{i\xi f(x)} d\xi.$$

If  $\varphi(x, \xi) = \xi f(x)$ , then since  $|\partial_x f(x)| = 1$   $\varphi$  is nondegenerate, and we may check that

$$C_\phi = M \times (\mathbb{R} \setminus \{0\}),$$

and

$$WF(A) = \Lambda_\phi = \{(x, \xi \partial_x f(x)) : x \in M\}.$$

This is nothing other than the normal bundle  $N^*M$  of  $M$ . ■

**Example 3.11** The oscillatory integral

$$A = \int_{\mathbb{R}^n} e^{i(\xi \cdot x - c|\xi|)} d\xi$$

arises naturally when we attempt to solve the acoustic wave equation using the Fourier transform. It is not difficult to check that  $\phi(x, \xi) = \xi \cdot x + c|\xi|$  is nondegenerate and

$$C_\phi = \{(x, \xi) : x = c\xi/|\xi|\} = \Lambda_\phi.$$

Thus

$$WF(A) = \{(c\omega, \lambda\omega) : \omega \in \mathbb{S}^{n-1}, \lambda \in \mathbb{R}^+\}.$$

■