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Regularity and multi-scale discretization of the solution construction of hyperbolic evolution equations with limited smoothness ☆

Maarten V. de Hoop [a], Sean F. Holman [a,*], Hart F. Smith [b], Gunther Uhlmann [b]

[a] Center for Computational and Applied Mathematics, and Geo-Mathematical Imaging Group, Purdue University, West Lafayette, IN 47907, USA
[b] Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA

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A B S T R A C T

We present a multi-scale solution scheme for hyperbolic evolution equations with curvelets. We assume, essentially, that the second-order derivatives of the symbol of the evolution operator are uniformly Lipschitz. The scheme is based on a solution construction introduced by Smith (1998) [1] and is composed of generating an approximate solution following a paradifferential decomposition of the mentioned symbol, here, with a second-order correction reminiscent of geometrical asymptotics involving a Hamilton–Jacobi system of equations and, subsequently, solving a particular Volterra equation. We analyze the regularity of the associated Volterra kernel, and then determine the optimal quadrature in the evolution parameter. Moreover, we provide an estimate for the spreading of (finite) sets of curvelets, enabling the multi-scale numerical computation with controlled error.

1. Introduction

We study the regularity in the construction of solutions of a general class of evolution equations with limited smoothness. We have applications to wave propagation in non-smooth media in mind. The construction makes use of a frame of curvelets [2–4], generates the weak solution on the one hand but reveals the geometrical properties reminiscent of the propagation of singularities in the case of smooth media on the other hand.

Let $p(z,x,\xi)$ be a real-valued function defined on $[0,Z]\times\mathbb{R}^n_x\times(\mathbb{R}^n_\xi\setminus\{0\})$ that is smooth and positively homogeneous of degree 1 with respect to ξ. If $z_0\in[0,Z]$, we then consider the initial value problem

$$\big(\partial_z - ip(z,x,D_x)\big)u(z,x)=0 \quad \text{and} \quad u(z_0,x)=u_0(x). \tag{1}$$

Here $p(z,x,D_x)$ is a pseudodifferential operator (ΨDO) whose symbol, p, may be rough in the z and x variables. We will require here that $p\in C^{m,1}S^1_{cl}$ with $m=1$ or 2. This means that the mth derivatives of p with respect to z and x exist everywhere, are uniformly Lipschitz, and for every multi-index α there is a constant C_α such that for all ξ sufficiently large

$$\big\|\partial_\xi^\alpha p(\cdot,\xi)\big\|_{C^{m,1}} \leqslant C_\alpha\big(1+|\xi|\big)^{1-|\alpha|}.$$

The notation of the left-hand side above indicates the $C^{m,1}$ norm in (z,x).

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* Corresponding author.
E-mail address: sfholman@math.purdue.edu (S.F. Holman).

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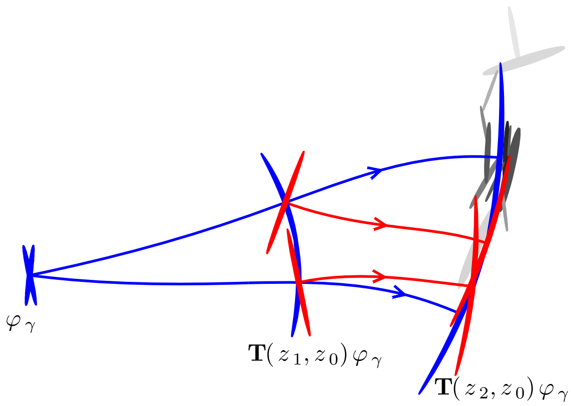

Fig. 1. Diagram describing a numeric solution including a single Volterra iteration. The parametrix is based on the second-order (geometric) approximation and depicted in blue. Some "scattered" curvelets produced by the Volterra iteration are depicted in red. The decay of curvelet frame coefficients for one of the wave packets propagated by the parametrix is illustrated by grayscale. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The technique used here for construction of solutions to (1) was introduced by Smith [1]. More recently, properties of these solutions were studied from the point of view of concentration of curvelets motivated by the propagation of singularities for the case of smooth symbols [5]. The solution construction is initiated by the construction of an approximate solution following the smoothing, that is, paradifferential decomposition of the symbol p, and is completed by solving a Volterra equation of the second kind which corrects for the symbol smoothing and essentially accounts for scattering between curvelets. The approximate solution is constructed using geometrical asymptotics and involves solving the Hamilton and Hamilton–Jacobi systems generated by the smoothed symbols. The Volterra equation can be solved by a Neumann series — as in the computation of certain multiple scattering series — revealing a curvelet–curvelet interaction (see Fig. 1). The main goal of this work is to develop regularity estimates in the evolution coordinate z for the Volterra kernel and solution. These estimates govern the choice of quadrature used when solving the Volterra equation, and subsequently the initial value problem, numerically.

Our main result uses an adapted underlying approximate solution operator (parametrix) for (1) with second-order correction. With this parametrix we provide scale-independent regularity estimates of the associated Volterra kernel in H^s, and likewise estimates for the regularity of the solution $g(z, x)$ of the Volterra equation in the z variable as a map into H^s, when $m = 2$ (or larger) and $-1 \leqslant s \leqslant 2$. Specifically, we obtain a Hölder estimate of order $1/2$. Thus a natural choice of quadrature when considering the numerical solution of the Volterra equation becomes the trapezoidal rule [6]. The approximate solution construction to second order is obtained from results pertaining to expansions of Fourier integral operators generated by canonical transformations [7,8]. This second-order parametrix improves on first-order parametrices in at least two ways. First, the Hölder regularity mentioned above is required to prove that a discretization of the Volterra equation in the evolution parameter z converges as the discretization step size goes to zero. Second, the Volterra kernel associated with the second-order parametrix is actually compact acting on H^s (in fact it maps into $H^{s+1/2}$) and so exhibits better behavior when iterated.

The results obtained here can be extended directly to apply to solving the second-order wave equation and associated Cauchy initial value problem.

A key aspect of developing an efficient computational algorithm will rely on available sparse decompositions of u_0 (that is, the initial data at z_0), and of the Volterra operator applied to the current solution by the Neumann series expansion (that is, the residual force at values of z dictated by the chosen quadrature). We have developed first steps towards an approach based on nonlinear approximation [9,10], motivated by the work of Beylkin and Monzón [11,12]. Here, we provide an estimate of the spreading of the set of curvelet coefficients under propagation as a function of scale. Following the decomposition of u_0 into wave packets, a natural solution strategy — tracing the convergence of the Neumann series expansion — starts at the finest available scale and progresses to the coarser scales. The Volterra equation can be solved with a step-by-step method reminiscent of the semi-group property. The numerical analysis of curvelet-like transforms can be found in [13,14]; this analysis plays a role in developing a fast algorithm for the above mentioned approximate solution. We note that the regularity and spreading estimates obtained here imply error estimates of corresponding numerical schemes. One possible such result is given in Corollary 12.

The results obtained in this paper have direct applications, for example, in seismic imaging. Indeed most imaging procedures can be expressed in terms of evolution equations [15]. We mention "reverse-time migration" based imaging [16] and "downward continuation (reverse depth)" based imaging [17,18]. Furthermore, curvelet based data regularization dovetails perfectly with these imaging techniques.

2. Solution of the evolution equation

We find solutions for (1) in two steps. We first construct an approximate solution operator, which we will refer to as a parametrix, and then we use this parametrix to transform (1) into an equivalent Volterra equation of the second kind for a function with values in a Sobolev space. To be more precise, we first construct a family of operators $\mathbf{T}(z,z') : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$ parametrized by $(z,z') \in [0,Z] \times [0,Z]$ which satisfy the following two properties

$$\mathbf{T}(z',z') = \mathrm{Id} \quad \text{for all } z' \in [0,Z], \quad \text{and} \tag{2}$$

$$\big(\partial_z - ip(z,x,D_x)\big)\mathbf{T}(z,z') : H^s \to H^s \quad \text{uniformly for } (z,z') \in [0,Z] \times [0,Z]. \tag{3}$$

Because $(\partial_z - ip(z,x,D_x))$ is a (possibly rough) ΨDO of order 1, the mapping property (3) is better than would be expected and so it is this property that makes $\mathbf{T}(z,z')$ an approximate solution operator. Here "uniformly" means that there is a single modulus of continuity that holds for all (z,z'). We think of $\mathbf{T}(z,z_0)u_0$ as giving the approximate solution of (1). Any family of operators satisfying these properties will be called a *parametrix*.

Once we have a parametrix we look for an exact solution for (1) in the form

$$u(z,x) = \big[\mathbf{T}(z,z_0)u_0\big](x) + \int_{z_0}^{z} \big[\mathbf{T}(z,z')g(z',\cdot)\big](x)\,\mathrm{d}z'. \tag{4}$$

Intuitively, we are setting $u(z,x)$ to be the approximate solution plus an error term that we expect can be found or at least estimated. Now the function to be determined is $g(z,x)$ which we refer to as the *residual*. A calculation making use of (2) shows that $u(z,x)$ is a solution of (1) if and only if

$$g(z,x) = -\big[\big(\partial_z - ip(z,x,D_x)\big)\mathbf{T}(z,z_0)u_0\big](x) - \int_{z_0}^{z} \big[\big(\partial_z - ip(z,x,D_x)\big)\mathbf{T}(z,z')g(z',\cdot)\big](x)\,\mathrm{d}z'.$$

Motivated by this fact we introduce the *Volterra kernel*

$$\mathbf{K}(z,z') = -\big(\partial_z - ip(z,x,D_x)\big)\mathbf{T}(z,z') \tag{5}$$

so that the equation for $g(z,x)$ becomes

$$g(z,x) = \big[\mathbf{K}(z,z_0)u_0\big](x) + \int_{z_0}^{z} \big[\mathbf{K}(z,z')g(z',\cdot)\big](x)\,\mathrm{d}z'. \tag{6}$$

This is a linear Volterra equation of the second kind where the function to be determined $(g(z,\cdot))$ takes values in the Sobolev space H^s. For a review of the classical theory of this type of equation see [6]. Although the theory there only explicitly deals with real and complex valued functions many of the results still hold in the case of functions valued in general Banach spaces with the same proofs.

The solution of (6) may be obtained via a Neumann series. Indeed, let us define

$$\mathbf{K}^1(z) = \mathbf{K}(z,z_0), \quad \text{and for } n > 1 \qquad \mathbf{K}^n(z) = \int_{z_0}^{z} \mathbf{K}(z,z')\mathbf{K}^{n-1}(z')\,\mathrm{d}z'.$$

Note that by (3) $\mathbf{K}(z,z') : H^s \to H^s$ uniformly, and thus the composition used in the iterative definition of $\mathbf{K}^n(z)$ is still an operator on H^s. Furthermore, if $\|\mathbf{K}(z,z')\|_{(H^s,H^s)} \leqslant C(Z)$ for all z and z', then for all n and z it follows from the definition that

$$\big\|\mathbf{K}^n(z)\big\|_{(H^s,H^s)} \leqslant \frac{Z^{n-1}}{(n-1)!}C(Z)^n. \tag{7}$$

The solution of (6) is then

$$g(z,x) = \sum_{n=1}^{\infty}\big[\mathbf{K}^n(z)u_0\big](x) =: \big[\mathbf{R}(z)u_0\big](x).$$

By (7) this sum converges absolutely in H^s for every $z \in [0,Z]$, and in fact

$$\big\|\mathbf{R}(z)\big\|_{(H^s,H^s)} \leqslant C(Z)e^{ZC(Z)}. \tag{8}$$

We refer to $\mathbf{R}(z)$ as the *resolvent* corresponding to the parametrix $\mathbf{T}$.

This method of solution was first introduced for the half wave equation in [1], and has been used previously to analyze equation (1) in [5]. In both of these works the parametrix $\mathbf{T}$ is constructed by decomposing u_0 in the curvelet frame, and then applying a rigid motion to each individual curvelet. We refer to this "rigid motion" parametrix as $\mathbf{T}_1$. In the current work we will introduce a new parametrix, $\mathbf{T}_2$, which still uses a curvelet decomposition of u_0, but also incorporates spreading into the evolution of each individual curvelet. As we will see, when $\mathbf{T}_2$ is used as the parametrix the corresponding Volterra kernel $\mathbf{K}_2(z,z')$ will have additional regularity properties in the z variables.

3. Construction of the parametrices

In this section we describe two possible ways to construct a parametrix satisfying the requirements (2) and (3). Both methods are based upon a curvelet decomposition. The first uses only a rigid motion of the curvelets, while the second also incorporates spreading. The treatment of caustics in the second method needs special attention, which we do not elaborate on here. The first method does not provide strong enough estimates to guarantee that numerical solutions of the Volterra equation will converge. The proofs in this section and Section 4 make use of the results in both of the appendices, and in particular the rules for manipulating families of curvelet like functions (FCLFs) developed in Appendix A. When dealing with an FCLF $\mathcal{F}$ we sometimes use the notation $f_\gamma \in \pi_{\mathcal{S}}(\mathcal{F})$ for a function in the family corresponding to the curvelet index γ.

The first step for both parametrices is to smooth the rough symbol p of (1) in the x variable according to scale. In this way we obtain a sequence of smooth (in x) symbols p_k which approximate p. Indeed, let $\psi \in C_c^\infty(\mathbb{R}^n)$ be an even function such that $\psi(\xi)=1$ for $|\xi|\leqslant 1$ and $\psi(\xi)=0$ for $|\xi|\geqslant 2$. We also assume $0\leqslant\psi\leqslant 1$ everywhere. Then define

$$p_k(z,x,\xi) = \left[\psi\left(2^{-k/2}D_x\right)p(z,\cdot_x,\xi)\right](x) \tag{9}$$

for all $k\in\mathbb{N}$. Thus for each k we low pass filter p in the x variable around the frequency $2^{k/2}$ to obtain $p_k\in C^\infty$. This sequence of approximations to p satisfies the following estimates. For $j+|\beta|\leqslant m+1$ (when $j+|\beta|=m+1$ estimate (10) holds everywhere the left-hand side is defined)

$$\left|\partial_z^j\partial_x^\beta\partial_\xi^\alpha(p-p_k)(z,x,\xi)\right| \lesssim 2^{-k(m-|\beta|-j+1)/2}\left\|\partial_\xi^\alpha p(\cdot,\xi)\right\|_{C^{m,1}} \tag{10}$$

and

$$\left|\partial_z^j\partial_x^\beta\partial_\xi^\alpha p_k(z,x,\xi)\right| \lesssim \left\|\partial_\xi^\alpha p(\cdot,\xi)\right\|_{C^{m,1}}. \tag{11}$$

Also, if $j\leqslant m$ and $|\beta|\geqslant m+1-j$ then

$$\left|\partial_z^j\partial_x^\beta\partial_\xi^\alpha p_k(z,x,\xi)\right| \lesssim 2^{k(|\beta|+j-m-1)/2}\left\|\partial_\xi^\alpha p(\cdot,\xi)\right\|_{C^{m,1}}. \tag{12}$$

Here and in the remainder of this work the notation $A_k\lesssim B_k$ means that there exists a constant $C>0$ independent of the scale k, or more generally the index $\gamma=(x,\nu,k)$, such that $A_k\leqslant CB_k$. In the following parametrix constructions $p_k(z,x,D_x)$ will be used to approximate the action of $p(z,x,D_x)$ on the curvelets at scale k.

3.1. Rigid motion parametrix

We first review the construction from [5] of a parametrix, referred to here as $\mathbf{T}_1(z,z')$, based only on the rigid motion of curvelets. The purpose of this review is twofold. First, some of the techniques involved will be used again in the construction of the new parametrix in Section 3.2, and second we eventually wish to compare some results for this parametrix and associated Volterra kernel to those for the new parametrix. In this interest we will also prove regularity estimates for $\mathbf{T}_1(z,z')$ in the z and z' variables. We will always assume that $m=1$ when we are considering $\mathbf{T}_1$.

We begin by considering the system

$$\begin{aligned}\frac{\mathrm{d}y_k}{\mathrm{d}z}\left(z,z'\right) &= -\partial_\xi p_k(z,y_k,\nu_k),\\ \frac{\mathrm{d}\nu_k}{\mathrm{d}z}\left(z,z'\right) &= \partial_x p_k(z,y_k,\nu_k)-\left\langle\nu_k,\partial_x p_k(z,y_k,\nu_k)\right\rangle\nu_k,\end{aligned} \tag{13}$$

and

$$\frac{\mathrm{d}\Theta_k}{\mathrm{d}z}\left(z,z'\right) = \Theta_k\left[\nu_k\otimes\partial_x p_k(z,y_k,\nu_k)-\partial_x p_k(z,y_k,\nu_k)\otimes\nu_k\right], \tag{14}$$

which gives the co-sphere projected Hamiltonian flow associated to p_k. We write

$$y_k\left(z,z',x,\nu\right),\qquad \nu_k\left(z,z',x,\nu\right),\quad\text{and}\quad \Theta_k\left(z,z',x,\nu\right)$$

for the solution of (13) and (14) with initial data

$$y_k\left(z',z',x,\nu\right)=x,\qquad \nu_k\left(z',z',x,\nu\right)=\nu,\quad\text{and}\quad \Theta_k\left(z',z',x,\nu\right)=\mathrm{Id},$$

and refer to the map $(x,\nu) \mapsto (y_k(z,z',x,\nu), \nu_k(z,z',x,\nu))$ as $\Psi^k_{z,z'}$. We also consider the system (13) and (14) with p_k replaced by p, and introduce a corresponding map $\Psi_{z,z'}$ defined in the analogous manner (note that since $p \in C^{1,1}$ the problem is well-posed). In [1] it is shown that

$$d\big(\Psi_{z,z'}(x,\nu), \Psi^k_{z,z'}(x,\nu)\big) \lesssim 2^{-k}, \tag{15}$$

where d is the pseudodistance defined in Appendix A.

If $\gamma = (x,\nu,k)$ is a curvelet index, then the *flow out* of the individual curvelet φ_γ is given by

$$\varphi_{1,\gamma}(z,z',y) = \varphi_\gamma\big(\Theta_k(z,z',x,\nu)\big(y - y_k(z,z',x,\nu)\big) + x\big).$$

If $u \in L^2(\mathbb{R}^n)$, then the parametrix $\mathbf{T}_1(z,z')$ is defined by

$$\big[\mathbf{T}_1(z,z')u\big](y) = \sum_\gamma u_\gamma \varphi_{1,\gamma}(z,z',y), \tag{16}$$

where the u_γ are the coefficients of u given by the curvelet co-frame. Since it will be useful below, we also define operators $\mathbf{T}_1^{k'}(z,z')$ which only give the contributions of curvelets at scale k':

$$\big[\mathbf{T}_1^{k'}(z,z')u\big](y) = \sum_{\{\gamma=(x,\nu,k):\ k=k'\}} u_\gamma \varphi_{1,\gamma}(z,z',y). \tag{17}$$

It is proven in [5] that $\mathbf{T}_1(z,z')$ is a parametrix as defined in Section 2 for $-1 \leqslant s \leqslant 2$.

Remark 1. We comment here that it should be possible in (16) to use elements of an FCLFs that also form a frame to define an operator similar to $\mathbf{T}_1$, but with respect to this alternate frame. The same comment applies later to the operator $\mathbf{T}_2$ introduced in the next section. Furthermore, essentially the same analysis should apply to that case. We remark that the frame of wave atoms, which also can be used to sparsely represent wave propagators, is not an FCLF, due to its finer frequency localization in the radial direction. The analogue of the parametrix $\mathbf{T}_2$ in the wave atom frame is the Gaussian beam approximation. (For information on wave atoms and their application to represent wave propagators see [19] and [20].)

To finish this section we prove the following regularity result for $\mathbf{T}_1(z,z')$.

Lemma 2. *The operator $\mathbf{T}_1(z,z')$ is uniformly Lipschitz in both of its arguments as a map from H^s to H^{s-1} for any s and on any fixed domain $[0,Z]\times[0,Z]$. That is*

$$\big\|\mathbf{T}_1(z,z') - \mathbf{T}_1(\bar z,z')\big\|_{(H^s,H^{s-1})} \leqslant C_1(Z)|z-\bar z| \tag{18}$$

for all z, $\bar z$, and $z' \in [0,Z]$, and the same estimate holds when z' is varied rather than z. Furthermore,

$$\big\|\mathbf{T}_1^{k'}(z,z') - \mathbf{T}_1^{k'}(\bar z,z')\big\|_{(L^2,L^2)} \leqslant C_1(Z)2^{k'}|z-\bar z| \tag{19}$$

and the same holds when z' is varied instead of z.

Proof. For $\gamma = (x_0,\nu^0,k)$ the index of a curvelet, we define the change of variables

$$y \mapsto \Phi_{1,\gamma}(z,z',y) := \Theta_k(z,z',x_0,\nu^0)\big(y - y_k(z,z',x_0,\nu^0)\big) + x_0.$$

This family of maps satisfies the hypotheses of Lemma 19 relative to the FCLF given by the curvelet frame. Therefore

$$\big\{2^{-k}\partial_z\varphi_{1,\gamma}(z,z',y), \Phi^*_{1,\gamma}(\gamma)\big\}$$

is an FCLF, and by Lemma 14 we have for every $\delta > 0$ a constant C_δ such that

$$\big|\big\langle\varphi_{\tilde\gamma}, \partial_z\varphi_{1,\gamma}(z,z',y)\big\rangle\big| \leqslant C_\delta 2^k \mu_\delta\big(\tilde\gamma, \Phi^*_{1,\gamma}(\gamma)\big), \tag{20}$$

where μ_δ is the weight function introduced in Appendix A. This is equivalent to

$$\big|\partial_z c_{1,\tilde\gamma\gamma}(z,z')\big| \leqslant C_\delta 2^k \mu_\delta\big(\tilde\gamma, \Psi^k_{z,z'}(\gamma)\big), \tag{21}$$

where $c_{1,\tilde\gamma,\gamma}(z,z')$ is the matrix for $\mathbf{T}_1$ with respect to the curvelet frame given by

$$c_{1,\tilde\gamma\gamma}(z,z') = \big\langle\varphi_{\tilde\gamma}(y), \varphi_{1,\gamma}(z,z',y)\big\rangle. \tag{22}$$

Using (15) we may replace $\Psi^k_{z,z'}$ by $\Psi_{z,z'}$ in (21) and so results in [1] imply that $\partial_z \mathbf{T}_1(z,z') : H^s \to H^{s-1}$ is uniformly bounded for all z and $z' \in [0,Z]$. Therefore

$$\left\|\mathbf{T}_1\left(z,z'\right) - \mathbf{T}_1\left(\bar{z},z'\right)\right\|_{(H^s,H^{s-1})} \leqslant \int_z^{\bar{z}} \left\|\partial_t \mathbf{T}_1\left(t,z'\right)\right\|_{(H^s,H^{s-1})} \mathrm{d}t \leqslant C_1(Z)|z-\bar{z}|,$$

where $C_1(Z) = \sup_{z,z'\in[0,Z]} \|\partial_z \mathbf{T}_1(z,z')\|_{(H^s,H^{s-1})}$. This proves (18). If we note that (21) implies

$$\left|\partial_z c^{k'}_{1,\tilde{\gamma}\gamma}\left(z,z'\right)\right| \leqslant C_\delta 2^{k'} \mu_\delta\left(\tilde{\gamma}, \Psi^{k'}_{z,z'}(\gamma)\right),$$

then (19) follows in the same way.

Finally, the result for the z' variable follows by the same proof if we begin by obtaining (20) where the differentiation is with respect to z' instead of z. □

Remark 3. We observe that both $\mathbf{T}^k_1(z,z')$ and $\mathbf{T}^k_1(z,z') - \mathbf{T}^k_1(\bar{z},z')$ are families of operators satisfying the hypotheses required for F_k in Lemma 21 with, respectively, $r=0$ and $r=1$ and in the latter case with $C = C_1(Z)|z-\bar{z}|$.

At this point we note that Lemma 2 together with the fact that $\mathbf{T}_1$ is a parametrix allow us to prove a (already known, see e.g. [21]) regularity result for the solution of (1). Indeed, from (4) we see that if $m=1$, then for $-1 \leqslant s \leqslant 2$ and initial data $u_0 \in H^s$ the solution $u(z,x)$ of (1) is in $C^{0,1}([0,Z];H^{s-1})$.

3.2. Parametrix with second-order correction

In this section we will construct a parametrix, $\mathbf{T}_2(z,z')$, that takes into account the spreading of curvelets. The action of this parametrix will be specified in the same way as in Section 3.1 by defining an action on each curvelet individually. The underlying motivation for the parametrix construction comes from an approximation to a Fourier integral operator, via a phase expansion as discussed in [7], with phase function defining the propagation of singularities for (1). In contrast to the parametrix $\mathbf{T}_1$ introduced in the previous section, $\mathbf{T}_2$ accounts for the full ray geometry, rather than just the rigid motion along a single ray, the natural spreading of the curvelets which occurs as they propagate, and a small phase shift. These are the effects that are necessary to obtain the next level of accuracy in an asymptotic solution for (1).

We should note that this parametrix construction only works in the absence of caustics (this restriction will be made more precise below). However, if there is a global minimum time before any caustics develop, then it is possible to repeatedly apply the construction stepping forward in sufficiently small time steps. Thus, though we do not formulate the precise statements here, these results can also apply past caustics. When we consider $\mathbf{T}_2(z,z')$ we will always assume that $m=2$.

We begin the construction by introducing the Hamiltonian system that gives the propagation of singularities for (1). In contrast to (13) and (14) the integral curves here are not projected onto the unit co-sphere. For every $(x,\eta) \in \mathbb{R}^n_x \times (\mathbb{R}^n_\eta \setminus \{0\})$ we consider the flow given by

$$\frac{\mathrm{d}y_k}{\mathrm{d}z}\left(z,z',x,\eta\right) = -\partial_\xi p_k(z,y_k,\nu_k) \quad \text{and} \quad \frac{\mathrm{d}\nu_k}{\mathrm{d}z}\left(z,z',x,\eta\right) = \partial_x p_k(z,y_k,\nu_k) \tag{23}$$

with initial data $y_k(z',z',x,\eta) = x$ and $\nu_k(z',z',x,\eta) = \eta$. The curves

$$\left(y_k\left(z,z',x,\eta\right), \nu_k\left(z,z',x,\eta\right)\right)$$

are the integral curves of the z-dependent Hamiltonian vector field given by p_k with initial data (x,η). We consider the following system of equations

$$(y,\nu) = \left(y_k\left(z,z',x,\eta\right), \nu_k\left(z,z',x,\eta\right)\right). \tag{24}$$

For every k these define a mapping from (z,z',x,η) to (y,ν) which is the canonical relation of the solution operator for (1) if p is replaced by p_k. Using them to define implicit relations amongst the various variables amounts to parametrizing this canonical relation by different subsets of the variables.

Now we supplement the flow (23) with another system that gives the dependence of (y,ν) on perturbations of (x,η). This system, the linearized Hamilton–Jacobi system associated to p_k, is

$$\frac{\mathrm{d}}{\mathrm{d}z}\,{}_kW\left(z,z',x,\eta\right) = \begin{pmatrix} -\partial^2_{\xi x} p_k(z,y_k,\nu_k) & -\partial^2_{\xi\xi} p_k(z,y_k,\nu_k) \\ \partial^2_{xx} p_k(z,y_k,\nu_k) & \partial^2_{x\xi} p_k(z,y_k,\nu_k) \end{pmatrix} {}_kW\left(z,z',x,\eta\right), \tag{25}$$

where ${}_kW(z,z',x,\eta)$ is a $2n \times 2n$ matrix with initial data ${}_kW(z',z',x,\eta) = \mathrm{Id}_{2n}$. We split ${}_kW(z,z',x,\eta)$ up into four $n \times n$ matrices

$${}_kW\left(z,z',x,\eta\right) = \begin{pmatrix} {}_kW_1(z,z',x,\eta) & {}_kW_2(z,z',x,\eta) \\ {}_kW_3(z,z',x,\eta) & {}_kW_4(z,z',x,\eta) \end{pmatrix},$$

and then we have

$$\frac{\partial y_k}{\partial x}\big(z,z',x,\eta\big) = {}_kW_1\big(z,z',x,\eta\big).$$

We will assume that ${}_kW_1(z,z',x,\eta)$ is always invertible, and so by the implicit function theorem Eq. (24) can be solved for x and ν as a function of (z,z',y,η) at least locally. Since these functions depend on k we will label them as $\tilde{x}_k$ and $\tilde{\nu}_k$. We can then introduce a defining function $S_k(z,z',y,\eta)$ for the canonical relation defined by (24) given by

$$S_k\big(z,z',y,\eta\big) = \big\langle \tilde{x}_k\big(z,z',y,\eta\big),\eta\big\rangle.$$

We will always assume that this map $\tilde{x}_k$ exists globally for z and $z' \in [0,Z]$. This is the assumption that there are no caustics. We can also find formulas for the derivatives of $\tilde{x}_k(z,z',y,\eta)$ and $\tilde{\nu}_k(z,z',y,\eta)$. In the following the matrices ${}_kW_i$ are understood to be evaluated at the point $(z,z',\tilde{x}_k(z,z',y,\eta),\eta)$.

$$\frac{\partial \tilde{x}_k}{\partial y}\big(z,z',y,\eta\big) = {}_kW_1^{-1}, \tag{26}$$

$$\frac{\partial \tilde{x}_k}{\partial \eta}\big(z,z',y,\eta\big) = -{}_kW_1^{-1}\,{}_kW_2, \tag{27}$$

$$\frac{\partial \tilde{\nu}_k}{\partial y}\big(z,z',y,\eta\big) = {}_kW_3\,{}_kW_1^{-1}, \tag{28}$$

and

$$\frac{\partial \tilde{\nu}_k}{\partial \eta}\big(z,z',y,\eta\big) = {}_kW_4 - {}_kW_3\,{}_kW_1^{-1}\,{}_kW_2. \tag{29}$$

Using the homogeneity of p_k we can also prove the two following properties

$$\eta = {}_kW_1^T\,\tilde{\nu}_k\big(z,z',y,\eta\big) \tag{30}$$

and

$${}_kW_2^T\,{}_kW_1^{-T}\eta = 0. \tag{31}$$

Here ${}_kW_1^{-T}$ refers to the inverse of the transpose of ${}_kW_1$. Finally, since $S_k(z,z',y,\eta)$ is a smooth function, using the above properties and the equality of the mixed partials of S_k we have

$${}_kW_1^{-T} = \left(\frac{\partial x_k}{\partial y}\right)^T = \left(\frac{\partial^2 S_k}{\partial y \partial \eta}\right)^T = \frac{\partial^2 S_k}{\partial \eta \partial y} = \frac{\partial \nu_k}{\partial \eta} = {}_kW_4 - {}_kW_3\,{}_kW_1^{-1}\,{}_kW_2. \tag{32}$$

Thus (29) becomes

$$\frac{\partial \tilde{\nu}_k}{\partial \eta}\big(z,z',y,\eta\big) = {}_kW_1^{-T}. \tag{29$'$}$$

Finally, we can check using some of the above identities that ${}_kW_3\,{}_kW_1^{-1}$ and ${}_kW_1^{-1}\,{}_kW_2$ are always symmetric matrices.

Next we will introduce the phase function used to construct the action of our parametrix on curvelets at scale k. Let $\gamma = (x_0,\nu^0,k)$ be the index of a curvelet. Then, for $z,z' \in [0,Z]$, $y \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^n$, define (motivated in part by an expansion of $S_k(z,z',y,\eta)$)

$$\begin{aligned}\tilde{S}_\gamma\big(z,z',y,\eta\big) = {}&\big\langle \tilde{x}_k\big(z,z',y,\nu^0\big),\eta\big\rangle - \frac{1}{2\langle \nu^0,\eta\rangle}\big\langle {}_kW_1^{-1}\,{}_kW_2\eta,\eta\big\rangle \\ &- \frac{1}{2}\int_{z'}^{z} \operatorname{tr}\big({}_kW_3\,{}_kW_1^{-1}\partial^2_{\xi\xi}p_k(t,y_k,\nu_k)\big)\,\mathrm{d}t,\end{aligned} \tag{33}$$

where the ${}_kW_i$, y_k, and ν_k are the functions defined above all evaluated at the point (t,z',x_0,ν^0) within the integrand and (z,z',x_0,ν^0) outside the integral. This will be the convention for the remainder of this work when ${}_kW_i$, y_k, or ν_k are written without any argument. Note that the last term in the definition only depends on z, z' and the curvelet index γ. Because of this we introduce the notation

$$U_\gamma\big(z,z'\big) = \frac{1}{2}\int_{z'}^{z} \operatorname{tr}\big({}_kW_3\,{}_kW_1^{-1}\partial^2_{\xi\xi}p_k(t,y_k,\nu_k)\big)\,\mathrm{d}t.$$

Now we define the action of an operator on the curvelet φ_γ as

$$\varphi_{2,\gamma}(z,z',y)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\tilde{S}_\gamma(z,z',y,\eta)}\hat{\varphi}_\gamma(\eta)\,\mathrm{d}\eta.$$

Note that $\varphi_{1,\gamma}$ could be written using the same formula if $\tilde{S}_\gamma$ were replaced by a linear phase function. To gain more intuition about the action of $\mathbf{T}_2$ we may consider the individual effects of each of the terms in the definition (33) of $\tilde{S}_\gamma$. The first term alone produces simply a change of variables that is done in accordance with the ray geometry. Thus $\mathbf{T}_2$ takes into account the full ray geometry rather than just rigid translations along individual rays as in the case of $\mathbf{T}_1$. The second term in (33) produces the spreading of the curvelets which naturally occurs as they propagate. Finally the third term in (33) produces a phase change along the rays which either advances of retards the phase in the direction of propagation.

Now $\mathbf{T}_2(z,z')$ and $\mathbf{T}_2^{k'}(z,z')$ are defined respectively by (16) and (17) with $\varphi_{1,\gamma}$ replaced by $\varphi_{2,\gamma}$. In Theorem 7 we will prove that $\mathbf{T}_2$ is a parametrix for (1), but for now we prove only the following analog of Lemma 2.

Lemma 4. *The results of Lemma 2 hold with $\mathbf{T}_1$ replaced by $\mathbf{T}_2$ assuming that no caustics develop in the interval* $[0,Z]$.

Proof. As one might suspect, the proof is similar to that of Lemma 2. We first note that

$$\partial_z\varphi_{2,\gamma}(z,z',y)=\frac{i}{(2\pi)^n}\int e^{i\tilde{S}_\gamma(z,z',y,\eta)}\partial_z\tilde{S}_\gamma(z,z',y,\eta)\hat{\varphi}_\gamma(\eta)\,\mathrm{d}\eta. \tag{34}$$

Now if we define new functions by

$$\hat{f}_\gamma(z,z',\eta)=e^{\frac{-i}{2\langle\nu^0,\eta\rangle}\langle{}_kW_1^{-1}{}_kW_2\eta,\eta\rangle}\hat{\varphi}_\gamma(\eta), \tag{35}$$

then using (31) we see that $\{f_\gamma(z,z',\cdot),\gamma\}_{\gamma\in\Gamma_0}$, where Γ_0 is the grid of indices corresponding to the curvelet frame, is an FCLF. Further, if we write

$$\Phi_{2,\gamma}(z,z',y)=\tilde{x}_k(z,z',y,\nu^0),$$

then from (34) we have

$$\partial_z\varphi_{2,\gamma}(z,z',y)=ie^{-iU_\gamma(z,z')}\big(\partial_z\tilde{S}_\gamma(z,z',y,D)f_\gamma(z,z',\cdot)\big)\big|_{\Phi_{2,\gamma}(z,z',y)}. \tag{36}$$

To simplify notation in the following we will write $y=\Phi_{2,\gamma}^{-1}(z,z',x)$ where $\Phi_{2,\gamma}^{-1}(z,z',\cdot)$ is the inverse of $y\mapsto\Phi_{2,\gamma}(z,z',y)$, and $\nu=\nu_k(z,z',x,\nu^0)$. From the definition of $\tilde{S}_\gamma$ as well as identities (26) and (32), we may calculate

$$\begin{aligned}\partial_z\tilde{S}_\gamma(z,z',y,\eta)&=\big\langle\partial_\xi p_k(z,y,\nu),\,{}_kW_1^{-T}(z,z',x,\nu^0)\eta\big\rangle+\frac{1}{2\langle\nu^0,\eta\rangle}\big\langle\partial^2_{\xi\xi}p_k(z,y_k,\nu_k)\,{}_kW_1^{-T}\eta,\,{}_kW_1^{-T}\eta\big\rangle\\&\quad-\frac{1}{2}\operatorname{tr}\big({}_kW_3\,{}_kW_1^{-1}\partial^2_{\xi\xi}p_k(z,y_k,\nu_k)\big).\end{aligned} \tag{37}$$

From this formula and using (12) we see that $\partial_z\tilde{S}_\gamma(z,z',\Phi_{2,\gamma}^{-1}(z,z',x),\eta)$ satisfies the hypotheses of Lemma 18, where z and z' are considered as parameters. By that lemma and Lemma 19,

$$\big\{2^{-k}\partial_z\varphi_{2,\gamma}(z,z',y),\Phi_{2,\gamma}^*(z,z',\cdot)(\gamma)\big\}_{\gamma\in\Gamma_0} \tag{38}$$

is an FCLF.

Now let us introduce the matrix coefficients $c_{2,\tilde{\gamma}\gamma}(z,z')$ defined by (22) with $\varphi_{1,\gamma}$ replaced by $\varphi_{2,\gamma}$. Just as in the proof of Lemma 2 from (38) it follows that

$$\big|\partial_z c_{2,\tilde{\gamma}\gamma}(z,z')\big|\leqslant C_\delta 2^k\mu_\delta\big(\tilde{\gamma},\Phi_{2,\gamma}^*(z,z',\cdot)(\gamma)\big)$$

for any $\delta>0$. Now we can calculate using (26) and (30)

$$\Phi_{2,\gamma}^*(z,z',\cdot)(\gamma)=\left(y_k(z,z',x_0,\nu^0),\frac{\nu_k(z,z',x_0,\nu^0)}{|\nu_k(z,z',x_0,\nu^0)|},k+\log_2\big(\big|\nu_k(z,z',x_0,\nu^0)\big|\big)\right).$$

The only difference between this and $\Phi_{1,\gamma}^*(z,z',\cdot)(\gamma)$ is the potential shift in the scale k by $\log_2(|\nu_k(z,z',x_0,\nu^0)|)$. However the size of this shift can be bounded uniformly by a constant times $Z\sup_{\omega\in\mathbb{S}^{n-1}}\|p(\cdot,\omega)\|_{C^{m,1}}$, and so as before we may replace $\Phi_{2,\gamma}^*(z,z',\cdot)(\gamma)$ by $\Psi_{z,z'}(\gamma)$. The results now follow as in the proof of Lemma 2. □

Remark 5. From the proof of Lemma 4, using Lemma 17, we may conclude that $\mathbf{T}_2^k(z,z')$ and $\mathbf{T}_2^k(z,z')-\mathbf{T}_2^k(\bar{z},z')$ satisfy the hypotheses for F_k in Lemma 21 with respectively $r=0$ and $r=1$, and in the latter case $C=C_2(Z)|z-\bar{z}|$. The constant A is related to the change in scale $\log_2(|\nu_k(z,z',x_0,\nu^0)|)$, giving the frequency localization.

4. Properties of the Volterra kernels and solutions

In this section we will prove a number of properties of the Volterra kernels $\mathbf{K}_1$ and $\mathbf{K}_2$ associated, by (5), respectively to the parametrices $\mathbf{T}_1$ and $\mathbf{T}_2$ introduced in the previous section.

4.1. Regularity estimates for the Volterra kernels

We will prove two theorems which give respectively Lipschitz and Hölder regularity estimates for $\mathbf{K}_1$ and $\mathbf{K}_2$. The key distinction is that $\mathbf{K}_2$ has Hölder regularity as a map from H^s to H^s for certain values of s, while $\mathbf{K}_1$ only has this type of regularity as a map from H^s to $H^{s-\epsilon}$ for some positive epsilon. Actually we will just prove a Lipschitz estimate for $\mathbf{K}_1$ as a map from H^s to H^{s-1}, but using interpolation such Hölder estimates could be found.

The first result concerns $\mathbf{K}_1$.

Theorem 6. *For* $-1/2 \leqslant s \leqslant 1$

$$\left\|\mathbf{K}_1\left(z, z'\right) - \mathbf{K}_1\left(\bar{z}, z'\right)\right\|_{(H^{s+1}, H^s)} \leqslant C_1^H(Z)|z-\bar{z}|, \tag{39}$$

uniformly in $z' \in [0, Z]$. *An equivalent estimate holds when* z' *is varied instead of* z.

Proof. To begin we make a decomposition of $(\partial_z - ip(z, y, D_y))\mathbf{T}_1(z, z') - (\partial_{\bar{z}} - ip(\bar{z}, y, D_y))\mathbf{T}_1(\bar{z}, z')$ into the following three pieces

$$\sum_k \left(\partial_z - ip_k(z, y, D_y)\right)\mathbf{T}_1^k\left(z, z'\right) - \left(\partial_{\bar{z}} - ip_k(\bar{z}, y, D_y)\right)\mathbf{T}_1^k\left(\bar{z}, z'\right), \tag{40}$$

$$i\sum_k \left(p_k(z, y, D_y) - p(z, y, D_y)\right)\left(\mathbf{T}_1^k\left(z, z'\right) - \mathbf{T}_1^k\left(\bar{z}, z'\right)\right), \tag{41}$$

and

$$i\sum_k \left(\left(p_k(z, y, D_y) - p_k(\bar{z}, y, D_y)\right) - \left(p(z, y, D_y) - p(\bar{z}, y, D_y)\right)\right)\mathbf{T}_1^k\left(\bar{z}, z'\right). \tag{42}$$

The proof that the kernel is Lipschitz in the z variable will be complete if we can estimate the norm of each of the previous three operators by $|\bar{z} - z|$. To estimate (41) we note, referring to Remark 3, that $\mathbf{T}_1^k(z, z') - \mathbf{T}_1^k(\bar{z}, z')$ satisfies the requirements of F_k in Lemma 21 with $r = 1$ and $C = C_1(Z)|z - \bar{z}|$. Also $p(z, y, \xi)$ takes the place of $a(y, \xi)$ with $m = 1$ for fixed z, and so Lemma 21 implies the required estimate for (41). Similarly, Lemma 21 implies the result for (42) taking this time $F_k = \mathbf{T}_1^k(\bar{z}, z')$, $r = 0$, $m = 0$, and $a(y, \xi) = p(z, y, \xi) - p(\bar{z}, y, \xi)$. We now continue to analyze (40).

We will use the same notation as in the proof of Lemma 2. First define

$$\tilde{\varphi}_{1,\gamma}\left(z, z', y\right) = \left(\partial_z - ip_k(z, y, D_y)\right)\varphi_{1,\gamma}\left(z, z', y\right) \tag{43}$$

and consider

$$\partial_z\tilde{\varphi}_{1,\gamma}\left(z, z', y\right) = \left(\partial_z - ip_k(z, y, D_y)\right)\partial_z\varphi_{1,\gamma}\left(z, z', y\right) - i\partial_z p_k(z, y, D_y)\varphi_{1,\gamma}\left(z, z', y\right). \tag{44}$$

Applying several of the lemmas from Appendix A to this formula we see that

$$\left\{2^{-k}\partial_z\tilde{\varphi}_{1,\gamma}\left(z, z', y\right), \Phi_{1,\gamma}^*\left(z, z', \cdot\right)(\gamma)\right\}_{\gamma\in\Gamma_0}$$

is an FCLF. Here we have omitted some calculations that show the cancellation of certain terms, but these calculations are essentially the same as some which can be found in [5], and a more sophisticated version is given in the proof of Theorem 7. The result for (40) now follows as in the proof of Lemma 2.

To prove the final statement about regularity in z' we write

$$\begin{aligned}\mathbf{K}_1\left(z, z'\right) - \mathbf{K}_1(z, \bar{z}) = {} & \sum_k \left(\partial_z - ip_k(z, y, D_y)\right)\left(\mathbf{T}_1^k\left(z, z'\right) - \mathbf{T}_1^k(z, \bar{z})\right) \\ & - i\sum_k \left(p_k(z, y, D_y) - p(z, y, D_y)\right)\left(\mathbf{T}_1^k\left(z, z'\right) - \mathbf{T}_1^k(z, \bar{z})\right).\end{aligned}$$

The required estimate for the first term in the sum above follows just as the estimate for (40), while the second term is estimated in the same way as (41). □

The following theorem regarding regularity of $\mathbf{K}_2$ is the main technical result of this paper.

Theorem 7. *For $-3/2 \leqslant s \leqslant 3$, $\mathbf{K}_2(z,z'): H^{s-1/2} \to H^s$ continuously. If $-1 \leqslant s \leqslant 2$ then*

$$\left\| \mathbf{K}_2(z,z') - \mathbf{K}_2(\bar{z},z') \right\|_{(H^s,H^s)} \leqslant C_2^H(Z)|z-\bar{z}|^{1/2}, \tag{45}$$

uniformly in $z' \in [0,Z]$. An equivalent estimate holds if we vary z' instead of z.

Remark 8. Note that the first statement of the theorem shows that $\mathbf{T}_2$ is a parametrix for $-3/2 \leqslant s \leqslant 3$.

Proof. First assume that $-3/2 \leqslant s \leqslant 3$. We begin as before by splitting $\mathbf{K}_2$ into smooth and rough parts:

$$\sum_k \bigl(\partial_z - i p_k(z,y,D_y)\bigr)\mathbf{T}_2^k + i \sum_k \bigl(p_k(z,y,D_y) - p(z,y,D_y)\bigr)\mathbf{T}_2^k.$$

For the rough part (the second summand above) we use the fact that $\mathbf{T}_2^k$ satisfies the requirements of Lemma 21 with $r=0$, and so the required estimates follow by applying the lemma with $m=2$. Now we continue to analyze the smooth part given by the first summand.

We use the same notation as in the proof of Lemma 4, and begin with formulas (36) and (37). Indeed, by (37) we have

$$\partial_z \tilde{S}_\gamma(z,z',y,D) f_\gamma = -i\bigl\langle \partial_\xi p_k(z,y,\nu),\ {}_kW_1^{-T}(z,z',x,\nu^0)\partial_x f_\gamma \bigr\rangle - \frac{1}{2}\operatorname{tr}\left({}_kW_1^{-1} \partial^2_{\xi\xi} p_k(z,y_k,\nu_k)\, {}_kW_1^{-T} \frac{\partial^2_{xx}}{\langle \nu^0, D\rangle} \right) f_\gamma$$
$$- \frac{1}{2}\operatorname{tr}\bigl({}_kW_3\, {}_kW_1^{-1} \partial^2_{\xi\xi} p_k(z,y_k,\nu_k)\bigr) f_\gamma.$$

Combined with (36) this gives a formula for $\partial_z \varphi_{2,\gamma}(z,z',y)$.

Next let us analyze $i p_k(z,y,D_y)\varphi_{2,\gamma}(z,z',y)$. Since

$$\varphi_{2,\gamma}(z,z',y) = e^{-iU_\gamma(z,z')}\bigl[\Phi^*_{2,\gamma}(z,z',\cdot) f_\gamma(z,z',\cdot)\bigr](y),$$

we may begin by applying the calculus of ΨDOs (see in particular [22, Theorem 18.1.17]) as well a generalization of [7, Lemma 3.1] to obtain the formula

$$p_k(z,y,D_y)\varphi_{2,\gamma}(z,z',y) = e^{-iU_\gamma(z,z')}\Phi^*_{2,\gamma}(z,z',\cdot)\bigl[g_\gamma(z,z',\cdot)\bigr](y),$$

where

$$g_\gamma(z,z',x) = \left(p_k\bigl(z,y,\mathrm{d}\Phi^T_{2,\gamma}(z,z',y)D\bigr) - \frac{1}{2}\operatorname{tr}\bigl({}_kW_3\, {}_kW_1^{-1}\partial^2_{\xi\xi} p_k(z,y_k,\nu_k)\bigr) + a_k(z,z',x,D) \right) f_\gamma$$

and the $a_k(z,z',x,\eta)$ result from the remainder terms of the ΨDO calculus and the application of the lemma. The symbols a_k are such that both

$$\bigl\{2^{k/2} a_k(z,z',x,D) f_\gamma(z,z',\cdot),\gamma\bigr\}_{\gamma\in\Gamma_0}$$

and

$$\bigl\{2^{-k/2} \partial_z a_k(z,z',x,D) f_\gamma(z,z',\cdot),\gamma\bigr\}_{\gamma\in\Gamma_0}$$

are families of curvelet like functions. This follows from analysis of these remainders and size estimates of ${}_kW_1$ and its derivatives. By Lemmas 20 and 16 if

$$\tilde{g}_\gamma = i2^{k/2} e^{-iU_\gamma(z,z')}\bigl(\partial_z \tilde{S}_\gamma(z,z',y,D) f_\gamma - g_\gamma\bigr) = i2^{k/2} e^{-iU_\gamma(z,z')} a_k(z,z',x,D) f_\gamma \tag{46}$$

then

$$\bigl\{\Phi^*_{2,\gamma}(z,z',\cdot)\tilde{g}_\gamma(z,z',\cdot),\ \Phi^*_{2,\gamma}(z,z',\cdot)(\gamma)\bigr\}_{\gamma\in\Gamma_0} \tag{47}$$

is an FCLF. Combining all the previous calculations we see that

$$2^{k/2}\bigl(\partial_z - i p_k(z,y,D_y)\bigr)\varphi_{2,\gamma}(z,z',y) = \bigl[\Phi^*_{2,\gamma}(z,z',\cdot)\tilde{g}_\gamma(z,z',\cdot)\bigr](y)$$

and we finally conclude that

$$\bigl\{2^{k/2}\bigl(\partial_z - i p_k(z,y,D_y)\bigr)\varphi_{2,\gamma}(z,z',y),\ \Phi^*_{2,\gamma}(z,z',\cdot)(\gamma)\bigr\}_{\gamma\in\Gamma_0} \tag{48}$$

is an FCLF. The first statement of the theorem now follows as in previous proofs.

To prove (45) we combine the result already obtained for the continuity of $\mathbf{K}_2(z,z')$ with the following estimate which we will show holds for $-3/2 \leqslant s \leqslant 2$.

$$\left\|\mathbf{K}_2\left(z,z'\right)-\mathbf{K}_2\left(\bar{z},z'\right)\right\|_{(H^{s+1/2},H^s)} \leqslant \frac{(C_2^H(Z))^2}{2}|z-\bar{z}| \tag{49}$$

for a constant $C_2^H(Z)>0$. Indeed, if we establish (49) then (45) follows by interpolation and the triangle inequality. The proof of (49) is the same as the proof of Theorem 6. First we split $\mathbf{K}_2(z,z')-\mathbf{K}_2(\bar{z},z')$ into (40)–(42) with $\mathbf{T}_1^k$ replaced by $\mathbf{T}_2^k$ wherever it appears. The estimates for the two rough parts, (41) and (42), follow just as before except that now $m=2$ and 1 in the respective applications of Lemma 21. Finally, we analyze the part corresponding to (40).

The object is to show that

$$\left\{2^{-k/2}\partial_z\left(\partial_z-ip_k(z,y,D)\right)\varphi_{2,\gamma}\left(z,z',y\right),\Phi_{2,\gamma}^*\left(z,z',\cdot\right)(\gamma)\right\}_{\gamma\in\Gamma_0} \tag{50}$$

is an FCLF from which (49) follows as in the previous proofs. To do this, we calculate using the results from above

$$2^{-k/2}\partial_z\left(\partial_z-ip_k(z,y,D)\right)\varphi_{2,\gamma}\left(z,z',y\right)=2^{-k}\left[\left(\partial_{\tilde{z}}\Phi_{2,\gamma}^*\left(\tilde{z},z',\cdot\right)\right)\tilde{g}_\gamma\left(z,z',\cdot\right)\big|_{\tilde{z}=z}+\Phi_{2,\gamma}^*\left(z,z',\cdot\right)\left(\partial_z\tilde{g}_\gamma\left(z,z',\cdot\right)\right)\right](y).$$

The first term on the right-hand side gives an FCLF by Lemma 19, and we can see that the second term also gives an FCLF by analyzing the derivative of the second line of (46). This completes the proof of (50) and also the proof of the Hölder regularity in z. To prove regularity in z' we begin with the same expression as (50) with the first ∂_z replaced by $\partial_{z'}$ and apply a similar analysis. □

4.2. *Estimates of the iterated Volterra kernel and solution*

Now that we have established our central technical results in the previous two sections, we apply them to the solution of the Volterra equation, and by extension the full solution of (1). Assume that we have a Volterra kernel $\mathbf{K}(z,z')$ with the following properties. There exist r, s, α, $C(Z)$, $C^H(Z)\in\mathbb{R}$ with $r\geqslant 0$, $1\geqslant\alpha>0$, and $C(Z)$, $C^H(Z)>0$ such that

$$\mathbf{K}\left(z,z'\right):H^s\to H^s\quad \text{uniformly for } z,z'\in[0,Z] \text{ with constant } C(Z), \tag{51}$$

$$\left\|\mathbf{K}\left(z,z'\right)-\mathbf{K}\left(\bar{z},\bar{z}'\right)\right\|_{(H^s,H^{s-r})}\leqslant C^H(Z)\left(|z-\bar{z}|^\alpha+\left|z'-\bar{z}'\right|^\alpha\right) \tag{52}$$

for all z, z', $\bar{z}$, and $\bar{z}'\in[0,Z]$. Note that the Volterra kernel associated to any parametrix satisfies (51), $\mathbf{K}_1$ from Section 3.1 satisfies (52) with certain values of s, $r=1$, and $\alpha=1$, and $\mathbf{K}_2$ from Section 3.2 satisfies (52) with certain values of s, $r=0$, and $\alpha=1/2$. Thus all the estimates of this section applied to either $\mathbf{K}_1$ or $\mathbf{K}_2$ may be considered as corollaries of Theorems 6 and 7.

We first consider the iterated Volterra kernel $\mathbf{K}^n$ given by (9). The following estimate is proven by applying (52) to the definition of the iterated kernel and using an inductive argument.

$$\left\|\mathbf{K}^n(z)-\mathbf{K}^n(\bar{z})\right\|_{(H^s,H^{s-r})}\leqslant\frac{Z^{n-1}}{(n-1)!}C(Z)^{n-1}C^H(Z)|z-\bar{z}|^\alpha. \tag{53}$$

We next consider the resolvent $\mathbf{R}(z)$ defined in Section 2 corresponding to $\mathbf{K}(z,z')$. By summing up (53) we obtain the following:

$$\left\|\mathbf{R}(z)-\mathbf{R}(\bar{z})\right\|_{(H^s,H^{s-r})}\leqslant e^{ZC(Z)}C^H(Z)|z-\bar{z}|^\alpha. \tag{54}$$

Now, for $u_0\in H^s$, let $g(z,x)=[\mathbf{R}(z)u_0](x)$ be the solution of the Volterra equation (6). A straightforward application of (54) then immediately implies that

$$\left\|g(z,\cdot)-g(\bar{z},\cdot)\right\|_{H^{s-r}}\leqslant e^{ZC(Z)}C^H(Z)|z-\bar{z}|^\alpha\|u_0\|_{H^s}. \tag{55}$$

This, together with the comments above, shows that if $0\leqslant s\leqslant 2$ and $g_1(z,x)$ is the solution of (6) using $\mathbf{K}_1$ with initial data $u_0\in H^s$, then $g_1\in C^{0,1}([0,Z];H^{s-1})$. If $-1\leqslant s\leqslant 2$ and $g_2(z,x)$ is the solution of (6) using $\mathbf{K}_2$ with initial data $u_0\in H^s$, then $g_2\in C^{0,1/2}([0,Z];H^s)$.

5. Approximation by semi-discretization

In this section we discretize the Volterra equation (6) with respect to the z variable. To accomplish this we use the repeated trapezoid rule to approximate the integral, and it is here that the regularity estimates from the previous section will play a key role. Using these estimates we have certain error bounds for the quadrature scheme which allow us to prove in turn convergence of a resulting approximation of the solution to the Volterra equation.

5.1. Quadrature scheme

To produce a numeric algorithm to solve the Volterra equation (6) we first introduce a quadrature scheme for the integration involved there. Given that the Volterra kernel and solution have Hölder regularity and in general no better, a natural choice of scheme is the trapezoid rule defined as follows.

For every $N \in \mathbb{N}$ we introduce a partition $\mathcal{P}^N = \{z_0^N, z_1^N, \dots, z_N^N\}$ of the interval $[0, Z]$. That is, we chose the z_i^N so that $0 = z_0^N < z_1^N < \cdots < z_N^N = Z$. We then define $h_i^N = z_i^N - z_{i-1}^N$, and the weights

$$w_{ij}^N = \begin{cases} \frac{h_j^N + h_{j+1}^N}{2} & \text{if } 0 < j < i \leqslant N, \\ \frac{h_j^N}{2} & \text{if } j = i > 0, \\ \frac{h_1^N}{2} & \text{if } j = 0 \text{ and } i > 0, \end{cases} \tag{56}$$

and otherwise $w_{ij}^N = 0$. Also let $h_N = \sup_{i \in 1,\dots,N} h_i^N$. If $\mathcal{B}$ is any Banach space and $f : [0, Z] \to \mathcal{B}$ is continuous, then the repeated trapezoid rule is given by

$$\int_0^{z_i^N} f(z)\,\mathrm{d}z = \sum_{j=0}^{i} w_{ij}^N f\big(z_j^N\big) + E_i^N(f). \tag{57}$$

We are thinking of the sum in the previous expression as an approximation to the integral and $E_i^N(f)$ as an error term which should approach zero as $N \to \infty$. Indeed, we have in general the following estimates

$$\big\|E_i^N(f)\big\|_{\mathcal{B}} \leqslant z_i^N (h_N)^\alpha \mathscr{L}^\alpha_{[0,z_i^N]}(f), \tag{58}$$

where

$$\mathscr{L}^\alpha_{[0,z_i^N]}(f) = \sup_{z,\bar{z} \in [0,z_i^N]: z \neq \bar{z}} \frac{\|f(z) - f(\bar{z})\|_{\mathcal{B}}}{|z - \bar{z}|^\alpha}.$$

The cases that are relevant here are when $f(z') = \mathbf{K}_m(z_i, z')g(z', \cdot) : [0, z_i] \to H^s$ where $m = 1$ or 2 and s is in the allowed range depending on m.

5.2. Semi-discrete Volterra equation

We now introduce the semi-discrete Volterra equation. If $\mathbf{K}$ is the Volterra kernel associated to a parametrix, as defined in Section 2, then we will write $\mathbf{K}_{ij}^N := \mathbf{K}(z_i^N, z_j^N)$. Also we define

$$A_{ij}^N := w_{ij}^N \mathbf{K}_{ij}^N,$$

where w_{ij} are the weights given in (56). Thus A^N is a $(N+1) \times (N+1)$ matrix with entries in the space of continuous linear operators from H^s to H^s. The *semi-discrete Volterra equation* is then

$$g_i^N = \mathbf{K}_{i0}^N u_0 + \sum_{j=0}^{i} A_{ij}^N g_j^N \quad \text{for all } i \in \{0, \dots, N\}. \tag{59}$$

We assume that $u_0 \in H^s$. If $h_N < 2$ then, since $A_{ii} = h_i \operatorname{Id}/2$ for $i > 0$, and $A_{00} = 0$, we see that this equation has a unique solution in $(H^s)^{N+1}$. This method of approximating g, using (59), is known as *direct quadrature*. The next proposition establishes how well the solution of (59) approximates the solution of the Volterra equation.

Proposition 9. *Suppose that the Volterra kernel $\mathbf{K}(z, z')$ satisfies (51) and (52) for some values of the parameters, and for given $u_0 \in H^s$ let $g(z, x) \in L^\infty([0, Z], H^s)$ be the solution of (6) and $g^N \in (H^s)^{N+1}$ be the solution of (59). Assume also that $\mathbf{K}(z, z')$ extends to a uniformly continuous map from H^{s-r} to H^{s-r} with the same constant $C(Z)$ and that $h_N < 1$. Then*

$$\sup_{i \in \{0,\dots,N\}} \big\|g(z_i, \cdot) - g_i^N(\cdot)\big\|_{H^{s-r}} \leqslant 2e^{(Z + \frac{2Nh_N}{2-h_N})C(Z)} C^H(Z) h_N^\alpha \|u_0\|_{H^s}. \tag{60}$$

Proof. We first note that by (8), (52), and (55)

$$\big\|\mathbf{K}\big(z_i^N, z'\big)g\big(z', \cdot\big) - \mathbf{K}\big(z_i^N, \bar{z}'\big)g\big(\bar{z}', \cdot\big)\big\|_{H^{s-r}} \leqslant 2e^{ZC(Z)} C^H(Z)\big|z' - \bar{z}'\big|^\alpha \|u_0\|_{H^s} \tag{61}$$

for all i, z', and $\bar{z}'$. We will write $d_i^N(x) = g(z_i^N, x) - g_i^N(x)$. Then (6) and (59) imply

$$d_i^N(x) = \left(\int_0^{z_i^N} \left[\mathbf{K}\left(z_i^N, z'\right) g\left(z', \cdot\right)\right](x)\,\mathrm{d}z' - \sum_{j=0}^{i} \left[A_{ij}^N g\left(z_j^N, \cdot\right)\right](x) \right) + \sum_{j=0}^{i} \left[A_{ij}^N \left(g\left(z_j^N, \cdot\right) - g_j^N(\cdot)\right)\right](x)$$
$$= E_i^N\left(\left[\mathbf{K}\left(z_i, z'\right) g\left(z', \cdot\right)\right]\right) + \sum_{j=0}^{i} \left[A_{ij}^N d_j^N(\cdot)\right](x). \tag{62}$$

Using now (58) with $\mathcal{B} = H^{s-r}$ and (61) we have

$$\left\|d_i^N\right\|_{H^{s-r}} \leqslant 2e^{ZC(Z)} C^H(Z) h_N^\alpha \|u_0\|_{H^s} + C(Z) \sum_{j=0}^{i} w_{ij}^N \left\|d_j^N\right\|_{H^{s-r}}$$
$$\leqslant 2\frac{e^{ZC(Z)} C^H(Z) h_N^\alpha \|u_0\|_{H^s}}{1 - h_N/2} + \frac{C(Z) h_N}{1 - h_N/2} \sum_{j=0}^{i-1} \left\|d_j^N\right\|_{H^{s-r}}.$$

A discrete Gronwall-type inequality (see [6, Section 1.5.3]) now implies that

$$\left\|d_i^N\right\|_{H^{s-r}} \leqslant 4e^{ZC(Z)} C^H(Z) h_N^\alpha \|u_0\|_{H^s} e^{\frac{N h_N C(Z)}{1 - h_N/2}}.$$

This completes the proof. □

If $h_N \approx 1/N$, as would be the case if the partition $\mathcal{P}$ is evenly spaced, then Proposition 9 implies that $\|g(z_i, \cdot) - g_i^N(\cdot)\|_{H^{r-s}} \approx (1/N)^\alpha$ as $N \to \infty$. The proposition also reveals the key difference between the parametrices $\mathbf{T}_1$ and $\mathbf{T}_2$ and corresponding Volterra kernels $\mathbf{K}_1$ and $\mathbf{K}_2$. For $\mathbf{K}_1$ we only have the Hölder estimates (52) in the case where $r > 0$, and so we can only estimate the error incurred as a result of the discretization in a norm which is rougher than that of the space where the initial data u_0 lies. However, for $\mathbf{K}_2$ we can take $r = 0$ and obtain an error estimate with respect to the original norm.

6. Concentration of sets of wave packets

In this section we assume that the initial data, u_0, has a representation in the curvelet frame that is concentrated near a finite set of curvelet indices Γ_0, and then study how much the curvelet coefficients for the solution of (1) spread away from $\Psi_{z,0}(\Gamma_0)$ as z increases. The motivation for this study is to apply the results to prove convergence of a numeric scheme to solve (1) using only a finite set of curvelets.

Following [5], we first introduce the following weighted spaces.

Definition 10. Let Γ_0 be a finite set of curvelet indices. We define the space $H_{\Gamma_0}^{\sigma,\alpha}$ by the norm

$$\|f\|_{H_{\Gamma_0}^{\sigma,\alpha}}^2 = \sum_{\gamma} \left| 2^{k\sigma} \min_{\gamma_0 \in \Gamma_0} \left\{ \left(2^{\max(k,k_0)} \bar{d}(\gamma; \gamma_0)\right)^\alpha \right\} f_\gamma \right|^2,$$

where $\gamma = (x, \nu, k)$, and f_γ are the coefficients of f with respect to the curvelet frame.

In this definition, σ corresponds to the Sobolev space regularity of f while α gives the degree to which the curvelet coefficients of f are concentrated near Γ_0. A useful estimate is the following

$$\|f\|_{H_{\Gamma_0}^{\sigma,\alpha}}^2 \approx \min_{\gamma_0 \in \Gamma_0} \sum_{\gamma} \left| 2^{k\sigma} \left(2^{\max(k,k_0)} \bar{d}(\gamma; \gamma_0)\right)^\alpha f_\gamma \right|^2. \tag{63}$$

The constants relating the two sides can be found based on the "radius" of the set Γ_0 (under a proper interpretation of the radius).

Estimates in terms of the $H_{\Gamma_0}^{\sigma,\alpha}$ norm allow us to easily estimate how well a given function is approximated by a finite set of curvelets. Indeed, given a finite set of curvelet indices $\Gamma_0 \subset \Gamma$ let Γ_0^r be the set of all indices γ that are indices of curvelets in the curvelet frame, and such that

$$\min_{\gamma_0 \in \Gamma_0} 2^{\max(k,k_0)} \bar{d}(\gamma; \gamma_0) \leqslant r.$$

Then define $\Delta^r_{\Gamma_0}$ to be the operator given by

$$\Delta^r_{\Gamma_0} f = \sum_{\gamma\in\Gamma_0^r} f_\gamma \varphi_\gamma.$$

If $\alpha \geqslant 0$ and $f \in H^{\sigma,\alpha}_{\Gamma_0}$ it is then straight forward to check that $f \in H^\sigma$ and

$$\left\| f - \Delta^r_{\Gamma_0} f \right\|_{H^\sigma} \leqslant \min\left(r^{-\alpha}, 1\right) \|f\|_{H^{\sigma,\alpha}_{\Gamma_0}}. \tag{64}$$

This inequality will be useful below when we estimate the error incurred by solving the Volterra equation with only a finite number of curvelets. However, to accomplish this goal we will first require the following lemma.

Lemma 11. *Let* $0 \leqslant \alpha < \frac{m+1}{2}$, $|\sigma| \leqslant \frac{m-1}{2}$ *with* $m = 1$ $(p \in C^{1,1}S^1_{cl})$ *or* 2 $(p \in C^{2,1}S^1_{cl})$. *It holds true that*

$$\left\| \mathbf{K}_m\left(z, z'\right) \right\|_{(H^{\sigma,\alpha}_{\Psi_{z',0}(\Gamma_0)}, H^{\sigma,\alpha}_{\Psi_{z,0}(\Gamma_0)})} \leqslant C_K(Z) \tag{65}$$

uniformly in $z, z' \in [0, Z]$.

Proof. Let $c_{m,\gamma\gamma'}(z,z')$ and $\tilde{c}_{m,\gamma\gamma'}(z,z')$ be respectively matrices of the operators

$$\sum_k \left(\partial_z - i p_k(z, y, D_y)\right)\mathbf{T}^k_m\left(z, z'\right) \quad \text{and} \quad \mathbf{T}_m\left(z, z'\right)$$

with respect to the curvelet frame. Then by results in Sections 3 and 4 as well as [1, Lemma 2.2] we have the estimates

$$\left| c_{m,\gamma\gamma'}\left(z, z'\right)\right| \lesssim 2^{k(1-m)/2} \mu_\delta\left(\gamma, \Psi_{z,z'}\left(\gamma'\right)\right) \quad \text{and} \quad \left|\tilde{c}_{m,\gamma\gamma'}\left(z, z'\right)\right| \lesssim \mu_\delta\left(\gamma, \Psi_{z,z'}\left(\gamma'\right)\right)$$

for any $\delta > 0$. Also, by [5, Theorem 5.5] (or, more accurately, using a portion of the proof of that theorem) and (63) we have the estimates

$$\left\| \left(p_k(z, y, D_y) - p(z, y, D_y)\right)\varphi_\gamma \right\|_{H^{\sigma,\alpha}_{\Psi_{z,0}(\Gamma_0)}} \lesssim \|\varphi_\gamma\|_{H^{\sigma,\alpha}_{\Psi_{z,0}(\Gamma_0)}} \lesssim 2^{k\sigma} \min_{\gamma_0 \in \Psi_{z,0}(\Gamma_0)} \left\{ \left(2^{\max(k,k_0)} \bar{d}(\gamma; \gamma_0)\right)^\alpha \right\}$$

for α and σ within the ranges specified in the hypotheses and where φ_γ is a curvelet at scale k. Making the same decomposition as in the proof of Theorem 7 we have

$$\begin{aligned} \left\| \mathbf{K}_m\left(z, z'\right) f \right\|^2_{H^{\sigma,\alpha}_{\Psi_{z,0}(\Gamma_0)}} \lesssim \Bigg(& \sum_\gamma \sum_{\gamma'} 2^{k\sigma} \min_{\gamma_0\in\Psi_{z,0}(\Gamma_0)} \left\{ \left(2^{\max(k,k_0)} \bar{d}(\gamma;\gamma_0)\right)^\alpha \right\} c_{m,\gamma\gamma'}\left(z, z'\right) |f_{\gamma'}| \\ & + \sum_\gamma \sum_{\gamma':k'=k} 2^{k\sigma} \min_{\gamma_0\in\Psi_{z,0}(\Gamma_0)} \left\{ \left(2^{\max(k,k_0)} \bar{d}(\gamma;\gamma_0)\right)^\alpha \right\} \tilde{c}_{m,\gamma\gamma'}\left(z, z'\right) |f_{\gamma'}| \Bigg)^2. \end{aligned}$$

Now we apply the estimate

$$\left(2^{\max(k,k_0)} \bar{d}\left(\gamma; \Psi_{z,0}(\gamma_0)\right)\right)^\alpha \lesssim \left(2^{\max(k,k')} \bar{d}\left(\gamma; \Psi_{z,z'}\left(\gamma'\right)\right)\right)^\alpha \cdot \left(2^{\max(k',k_0)} \bar{d}\left(\Psi_{z,0}(\gamma_0); \Psi_{z,z'}\left(\gamma'\right)\right)\right)^\alpha$$

which together with the bounds from above on the matrix coefficients $c_{m,\gamma\gamma'}$ and $\tilde{c}_{m,\gamma\gamma'}$ gives for any $\delta > 0$

$$\left\| \mathbf{K}_m\left(z, z'\right) f \right\|^2_{H^{\sigma,\alpha}_{\Psi_{z,0}(\Gamma_0)}} \lesssim \min_{\gamma_0\in\Psi_{z,0}(\Gamma_0)} \sum_\gamma \left(\sum_{\gamma'} \mu_\delta\left(\gamma_0, \Psi_{z,z'}\left(\gamma'\right)\right) \left| 2^{k'\sigma} \left(2^{\max(k',k_0)} \bar{d}\left(\gamma_0; \Psi_{z,z'}\left(\gamma'\right)\right)\right)^\alpha f_{\gamma'} \right| \right)^2.$$

Finally, [1, Lemmas 2.1, 2.2, 2.4] imply with the last inequality and (63) that

$$\left\| \mathbf{K}_m\left(z, z'\right) f \right\|^2_{H^{\sigma,\alpha}_{\Psi_{z,0}(\Gamma_0)}} \lesssim \|f\|^2_{H^{\sigma,\alpha}_{\Psi_{z',0}(\Gamma_0)}}.$$

This completes the proof. □

The lemma also yields estimates for the resolvents:

$$\left\| \mathbf{R}_m(z) \right\|_{(H^{\sigma,\alpha}_{\Gamma_0}, H^{\sigma,\alpha}_{\Psi_{z,0}(\Gamma_0)})} \leqslant C_{R,m}(Z) \tag{66}$$

uniformly for $z \in [0, Z]$.

With the previous result we may now prove an error estimate that relates the solution of the fully discrete Volterra equation (i.e. the semi-discrete equation from the previous section truncated to a finite set of curvelets) to the true Volterra solution. We begin by modifying the semi-discrete Volterra equation to become fully discrete. Using the same notation as in Section 5.2, for any given $r > 0$ and finite set of indices Γ_0 let

$$\tilde{A}_{ij}^N = w_{ij} \Delta^r_{\Psi_{z_i^N,0}(\Gamma_0)} \mathbf{K}_{ij}.$$

The *fully discrete Volterra equation* is then (compare with (59))

$$\tilde{g}_i^N = \Delta^r_{\Psi_{z_i^N,0}(\Gamma_0)} \mathbf{K}_{i0} u_0 + \sum_{j=0}^{i} \tilde{A}_{ij} \tilde{g}_j^N. \tag{67}$$

Note that for every i and N the solution $\tilde{g}_i^N$ of (67) is a linear combination of curvelets corresponding to the indices $\Psi_{z_i^N,0}(\Gamma_0)^r$. Now we present the result, which is a sort of extension of Proposition 9.

Corollary 12. *Let u_0 be a linear combination of curvelets with indices in the finite set Γ_0. Suppose that $g(z,x)$ is the solution of (6) corresponding to $\mathbf{K}_2$, and that $\tilde{g}^N$ is the corresponding solution of (67). Then for any $|s| \leqslant 1/2$ and $0 \leqslant \alpha < 3/2$ we have the estimate*

$$\sup_{i \in \{0,\ldots,N\}} \left\| g\left(z_i^N, \cdot\right) - \tilde{g}_i^N \right\|_{H^s} \leqslant C(Z)\left(h_N^{1/2} + \min\left(r^{-\alpha}, 1\right)\right) \|u_0\|_{H^s}.$$

Proof. The proof is largely the same as the proof of Proposition 9. The primary difference is that in (62) $\tilde{d}_i^N(x) = g(z_i^N, x) - \tilde{g}_i^N(x)$ replaces $d_i^N(x)$, $\tilde{A}_{ij}^N$ replaces A_{ij}^N in the sum on the second line, and there appear the extra terms

$$\left\| \sum_{j=0}^{i} \left[A_{ij}^N - \tilde{A}_{ij}^N\right] g\left(z_j^N, \cdot\right) \right\|_{H^s} \leqslant \min\left(r^{-\alpha}, 1\right) Z C_2(Z) C_{R,2}(Z) \|u_0\|_{H^s}$$

and

$$\left\| \mathbf{K}_{i0} u_0 - \Delta^r_{\Psi_{z_i^N,0}(\Gamma_0)} \mathbf{K}_{i0} u_0 \right\|_{H^s} \leqslant \min\left(r^{-\alpha}, 1\right) C_2(Z) \|u_0\|_{H^s}.$$

These estimates use the result of Lemma 11, (64), and (66) as well as the continuity of $\mathbf{K}_2$. Inserting these into the proof of Proposition 9 yields the proof of the corollary. □

This last corollary establishes the possibility of approximating the solution of the Volterra equation using only curvelets that lie within a certain distance of the Hamiltonian flow corresponding to the finite number of initial curvelets. We note additionally that the estimates lend themselves well to a “step-by-step” approach to solving the fully discrete Volterra equation (67). Given a choice of step size in the quadrature, and a choice for r, at each time step we compute only those curvelet coefficients corresponding to indices in $\Psi_{z_i^N,0}(\Gamma_0)^r$. This means, loosely speaking, that we only consider those curvelets lying within r of the original curvelets flowed forward to time z_i.

In the same vein as Remark 1 above, we comment that it should be possible to replace the frame of curvelets by another frame that is also an FCLF. Since the major part of our analysis leading up to the results in this section uses only the properties of FCLFs, it should then proceed in the same manner, and for a numerical scheme based on a parametrix constructed using this different frame we expect the same result as Corollary 12. We point out that in harmonic analysis there are a number of distinct “wave packet” frames and, speaking again somewhat loosely, amongst these those which are based on a parabolic scaling in phase space will generally be FCLFs.

Finally, we point out that the approximate solution of (1) provided by the fully discrete Volterra equation will consist of a sum of terms each being a composition of some number of operators of the form $\mathbf{K}_2(z_i^N, z_j^N)$ or $\mathbf{T}_2(z_i^N, z_j^N)$. We comment that there will be a further error which has not been analyzed here arising from the numerical computation of these operators which might be done in several ways. An efficient method might be carried out via a separated representation similar to that used in the proofs of Theorems 6 and 7. A full analysis of this is reserved for future work. A method of numerical implementation for $\mathbf{T}_2$ and some analysis of the associated error has been done in [23].

Appendix A. Curvelet like functions

In this appendix we develop some technical machinery which we use to analyze the various operators defined in terms of the curvelet frame. In the main text, we use a curvelet frame based on parabolic scaling as defined, for example, in either [5] or [1]. Our notation for curvelets and the curvelet frame matches that of [5]. In particular, we use the notation

$\Gamma = \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}$ and refer to Γ as the set of "curvelet indices." Also d is the pseudodistance on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ introduced in [24, Definition 2.1]

$$d(x,\nu;x',\nu') = \left|\langle \nu, x-x'\rangle\right| + \left|\langle \nu', x-x'\rangle\right| + \min\left\{|x-x'|, |x-x'|^2\right\} + |\nu-\nu'|^2. \tag{A.1}$$

If $\gamma = (x,\nu,k)$ and $\gamma' = (x',\nu',k') \in \Gamma$, let

$$\bar{d}(\gamma;\gamma') = 2^{-\min(k,k')} + d(x,\nu;x',\nu'). \tag{A.2}$$

The weight function $\mu_\delta(\gamma,\gamma')$ is given by

$$\mu_\delta(\gamma,\gamma') = \left(1+|k'-k|^2\right)^{-1} 2^{-(\frac{1}{2}n+\delta)|k'-k|} 2^{-(n+\delta)\min(k',k)} \bar{d}(\gamma;\gamma')^{-(n+\delta)}.$$

This weight function is different from, but equivalent to that introduced in [24]. We also use both notations $\hat{f}$ and $\mathscr{F}\{f\}$ for the Fourier transform of f depending on the aesthetic demands of the individual situation.

The curvelets at scale zero require a brief special note. These elements of the frame do not have a direction and so are indexed only by their position. Nonetheless in sums over the frame such as (16) and (63) we include these zero scale curvelets without comment. If $\gamma = (x,0)$ is the index of a zero scale curvelet then the function $\bar{d}$ defined in the previous paragraph is modified to

$$\bar{d}(\gamma;\gamma') = 1 + |x-x'|^2$$

and this is then used in the definition of the weights u_δ when one of the indices is at the zero scale.

We now begin to introduce more general classes of functions that behave in many ways like those which make up the curvelet frame. For $k \in \mathbb{R}$ we will denote by C_k the cylinder

$$C_k = \left[2^{k-1}, 2^{k+1}\right] \times \mathbb{B}^{n-1}_{2^{k/2}} \subset \mathbb{R}^n, \tag{A.3}$$

where $\mathbb{B}^{n-1}_{2^{k/2}}$ is the $(n-1)$-dimensional ball of radius $2^{k/2}$ centered at the origin. The term "dyadic parabolic scaling" refers to the relative proportions of these cylinders which scale like 2^k in the direction of e_1, and $2^{k/2}$ in the perpendicular directions. Given $\nu \in \mathbb{S}^{n-1}$ let $\Theta_\nu \in O(n)$ represent any rotation that maps e_1 into ν, and define

$$C_{\nu,k} = \Theta_\nu C_k.$$

Naturally $C_{\nu,k}$ is independent of the specific rotation that is chosen. Also, we write $\rho_k = |C_k| \sim 2^{k(n+1)/2}$. The families of functions are now defined as follows.

Definition 13. A subset $\mathcal{F} \subset \mathcal{S}(\mathbb{R}^n) \times \Gamma$ is a **family of curvelet like functions (FCLF)** if the following conditions are met:

1. For every $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $N \in \mathbb{N}$, there exists a constant $C_{j,\alpha,N} > 0$ such that the following estimates hold for all $(f,(x,\nu,k)) \in \mathcal{F}$

$$\rho_k^{1/2}\left|\langle \nu, \partial_\xi\rangle^j \partial_\xi^\alpha\left(e^{i\langle x,\xi\rangle}\hat{f}(\xi)\right)\right| \leqslant C_{j,\alpha,N} 2^{-k(j+\frac{|\alpha|}{2})}\left(1+2^{-k/2}\|\xi - C_{\nu,k}\|\right)^{-N}.$$

2. There exists a constant $C \in \mathbb{R}$ (possibly less than zero) such that for all $(f,(x,\nu,k)) \in \mathcal{F}$, $k \geqslant C$.

When we have a family of curvelet like functions, $\mathcal{F}$, we use the notation $\pi_\Gamma : \mathcal{F} \to \Gamma$ for the map projecting $\mathcal{F}$ onto the set of curvelet indices, and π_x, π_ν, π_k for the respective projections onto components of the curvelet indices. Also, $\pi_{\mathcal{S}} : \mathcal{F} \to \mathcal{S}(\mathbb{R}^n)$ is the projection onto $\mathcal{S}(\mathbb{R}^n)$. When referring to a fixed family of curvelet like functions we will usually write $\gamma = (x,\nu,k)$ for the curvelet index of arbitrary functions in the family.

We make the observation that if $\mathcal{F}$ and $\mathcal{G}$ are FCLFs such that $\pi_\Gamma(\mathcal{F}) = \pi_\Gamma(\mathcal{G})$, then we may form another FCLF as $\mathcal{F}+\mathcal{G} = \{(f+g,\gamma)\colon (f,\gamma)\in\mathcal{F} \text{ and } (g,\gamma)\in\mathcal{G}\}$. That $\mathcal{F}+\mathcal{G}$ defined in this way is in fact an FCLF follows easily from the definition.

Curvelet frames with parabolic scaling give families of curvelet like functions if we remove the elements of the frame whose Fourier transform covers the origin (i.e. the zero scale curvelets). The motivation for considering these families is that they are more or less preserved under most of the operations that we would like to perform on curvelets. In the following series of lemmas we will show precisely what this means, and in essence establish a calculus for families of curvelet like functions.

Lemma 14. *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two families of curvelet like functions. Then for every $\delta > 0$ there exists a constant C_δ such that for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$*

$$\left|\langle \pi_{\mathcal{S}}(f), \pi_{\mathcal{S}}(g)\rangle\right| \leqslant C_\delta \mu_\delta\left(\pi_\Gamma(f), \pi_\Gamma(g)\right).$$

Proof. We first prove the result for the case when one of the families is given by a curvelet frame. Then we represent the functions in each of two families with respect to this curvelet frame and apply a slight generalization of [1, Lemma 2.5] to the case when the γ and γ_0 need not be in the grid corresponding to the curvelet frame. □

We next study what happens when we take derivatives of curvelet like functions.

Lemma 15. *Suppose that $\mathcal{F}$ is a family of curvelet like functions. Then*

$$\big\{\big(2^{-k}\langle\nu,\partial_y\rangle f,\gamma\big)\big\}_{(f,\gamma)\in\mathcal{F}}\quad\textit{and}\quad\big\{\big(2^{k}\langle\nu,y-x\rangle f,\gamma\big)\big\}_{(f,\gamma)\in\mathcal{F}}$$

are also families of curvelet like functions. Furthermore, if we are given a map $\nu_\perp:\mathcal{F}\to\mathbb{S}^{n-1}$ such that $\langle\nu_\perp(f,\gamma),\nu\rangle=0$ for every $(f,\gamma)\in\mathcal{F}$, then

$$\big\{\big(2^{-k/2}\big\langle\nu_\perp(f,\gamma),\partial_y\big\rangle f,\gamma\big)\big\}_{(f,\gamma)\in\mathcal{F}}$$

and

$$\big\{\big(2^{k/2}\big\langle\nu_\perp(f,\gamma),y-x\big\rangle f,\gamma\big)\big\}_{(f,\gamma)\in\mathcal{F}}$$

are both families of curvelet like functions.

Proof. For $(f,\gamma)\in\mathcal{F}$ we have

$$2^{-k}\mathscr{F}\big\{\langle\nu,\partial_y\rangle f\big\}(\xi)=i2^{-k}\langle\nu,\xi\rangle\hat{f}(\xi).$$

Combined with the inequality

$$2^{-k}\big|\langle\nu,\xi\rangle\big|\leqslant 2\big(1+2^{-k/2}\|\xi-C_{\nu,k}\|\big)$$

this gives the first assertion of the lemma.

Next we have

$$\mathscr{F}\big\{\langle\nu,y-x\rangle f\big\}=\langle\nu,D\hat{f}\rangle,$$

which easily implies the second assertion.

The third and fourth assertions follow in the same way if we use also the inequality

$$2^{-k/2}\big|\big\langle\nu_\perp(f,\gamma),\xi\big\rangle\big|\leqslant\big(1+2^{-k/2}\|\xi-C_{\nu,k}\|\big)$$

which holds for any $\nu_\perp$ satisfying the hypotheses. □

We next study how curvelet like functions change under pull-back by a change of coordinates. Suppose that $\Phi:\mathbb{R}^n\to\mathbb{R}^n$ is a diffeomorphism. First recall that the pull-back of a function $f\in\mathcal{S}(\mathbb{R}^n)$ is given by the composition $\Phi^*(f)(x)=f(\Phi(x))$. We define the pull-back of a curvelet index $\gamma=(x,\nu,k)$ by

$$\Phi^*(\gamma)=\big(\Phi^{-1}(x),\big(\mathrm{d}\Phi^T(x)\nu\big)/\big|\mathrm{d}\Phi^T(x)\nu\big|,k+\log_2\big(\big|\mathrm{d}\Phi^T(x)\nu\big|\big)\big).$$

Note that since Φ is a diffeomorphism, the map $\Phi^*:\Gamma\to\Gamma$ is invertible.

Lemma 16. *Suppose that $\mathcal{F}$ is a family of curvelet like functions, and that $\{\Phi_\gamma\}_{\gamma\in\pi_\Gamma(\mathcal{F})}$ is a family of diffeomorphisms on $\mathbb{R}^n$ satisfying*

$$\big\|\partial_x^\alpha\Phi_\gamma^{\pm1}\big\|\leqslant C_\alpha\quad\textit{for } 0<|\alpha|\leqslant 2\quad\textit{and}\quad\big\|\partial_x^\alpha\Phi_\gamma^{\pm1}\big\|\leqslant 2^{k(|\alpha|-2)/2}C_\alpha\quad\textit{for } 2<|\alpha|.$$

Then

$$\big\{\big(\Phi_\gamma^*(f),\Phi_\gamma^*(\gamma)\big)\big\}_{(f,\gamma)\in\mathcal{F}}$$

is a family of curvelet like functions. Note that $\Phi_\gamma^*(f)$ is the pull-back of the function f, while $\Phi_\gamma^*(\gamma)$ is the pull-back of the curvelet index γ.

Proof. Let $(f,\gamma)\in\mathcal{F}$. By the Fourier inversion formula we have the following formula

$$\begin{aligned}\big\langle\nu',\partial_\xi\big\rangle^j\partial_\xi^\alpha\big(e^{i\langle\Phi_\gamma^{-1}(x),\xi\rangle}\widehat{\Phi_\gamma^*f}(\xi)\big)={}&\frac{i^{j+|\alpha|}}{(2\pi)^n}\iint e^{i(\langle\Phi_\gamma(y)-x,\eta\rangle-\langle y-\Phi_\gamma^{-1}(x),\xi\rangle)}\\&\times\big\langle\nu',\Phi_\gamma^{-1}(x)-y\big\rangle^j\big(\Phi_\gamma^{-1}(x)-y\big)^\alpha\big(e^{i\langle x,\eta\rangle}\hat{f}(\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}y,\end{aligned}$$

where $k' \in \mathbb{R}$ and $\nu' \in \mathbb{S}^{n-1}$ are the respective components of the pull-back $\Phi_\gamma^*(\gamma)$. Note that this should be interpreted as an iterated integral with the integration done first in η and then in y. Making the change $\tilde{x} = \Phi_\gamma(y) - x$ in the second integral gives

$$\langle \nu', \partial_\xi \rangle^j \partial_\xi^\alpha \left(e^{i\langle \Phi_\gamma^{-1}(x), \xi\rangle} \widehat{\Phi_\gamma^* f}(\xi) \right) = \frac{i^{j+|\alpha|}}{(2\pi)^n} \iint e^{i(\langle \tilde{x}, \eta\rangle - \langle \Phi_\gamma^{-1}(\tilde{x}+x) - \Phi_\gamma^{-1}(x), \xi\rangle)} \langle \nu', \Phi_\gamma^{-1}(x) - \Phi_\gamma^{-1}(\tilde{x}+x)\rangle^j \times \left(\Phi_\gamma^{-1}(x) - \Phi_\gamma^{-1}(\tilde{x}+x)\right)^\alpha \left(e^{i\langle x, \eta\rangle} \hat{f}(\eta)\right) \frac{\mathrm{d}\eta \, \mathrm{d}\tilde{x}}{|\det(\mathrm{d}\Phi_\gamma(\tilde{x}+x))|}.$$

By Taylor's theorem we may write

$$\Phi_\gamma^{-1}(\tilde{x}+x)^r - \Phi_\gamma^{-1}(x)^r = \mathrm{d}\Phi_\gamma^{-1}(x)_p^r \tilde{x}^p + \tilde{\Psi}_\gamma(\tilde{x}, x)_{pq}^r \tilde{x}^q \tilde{x}^p$$

using the summation convention. Here $\tilde{\Psi}_\gamma$ is a smooth array of functions that can all be simultaneously bounded in C^l in terms of bounds on the derivatives of Φ_γ^{-1} up to order $l+2$. We will also write $\tilde{\Phi}_\gamma(\tilde{x}, x)_p^r = \mathrm{d}\Phi_\gamma^{-1}(x)_p^r + \tilde{\Psi}_\gamma(\tilde{x}, x)_{pq}^r \tilde{x}^q$. With this notation

$$\begin{aligned} \langle \nu', \partial_\xi \rangle^j \partial_\xi^\alpha \left(e^{i\langle \Phi_\gamma^{-1}(x), \xi\rangle} \widehat{\Phi_\gamma^* f}(\xi) \right) &= \frac{(-i)^{j+|\alpha|}}{(2\pi)^n} \iint e^{i(\langle \tilde{x}, \eta\rangle - \langle \tilde{\Phi}_\gamma(\tilde{x}, x)\tilde{x}, \xi\rangle)} \langle \nu', \tilde{\Phi}_\gamma(\tilde{x}, x)\tilde{x}\rangle^j \\ &\quad \times \left(\tilde{\Phi}_\gamma(\tilde{x}, x)\tilde{x}\right)^\alpha \left(e^{i\langle x, \eta\rangle} \hat{f}(\eta)\right) \frac{\mathrm{d}\eta \, \mathrm{d}\tilde{x}}{|\det(\mathrm{d}\Phi_\gamma(\tilde{x}+x))|} \\ &= \frac{(-i)^j}{(2\pi)^n} \iint e^{i(\langle \tilde{x}, \eta\rangle - \langle \tilde{\Phi}_\gamma(\tilde{x}, x)\tilde{x}, \xi\rangle)} \left(\langle \nu', \mathrm{d}\Phi_\gamma^{-1}(x)\partial_\eta\rangle + \left(\nu_r' \tilde{\Psi}_\gamma(\tilde{x}, x)_{pq}^r \partial_{\eta_p \eta_q}^2\right)\right)^j \\ &\quad \times \left(\tilde{\Phi}_\gamma(\tilde{x}, x)\partial_\eta\right)^\alpha \left(e^{i\langle x, \eta\rangle} \hat{f}(\eta)\right) \frac{\mathrm{d}\eta \, \mathrm{d}\tilde{x}}{|\det(\mathrm{d}\Phi_\gamma(\tilde{x}+x))|}. \end{aligned}$$

Now if we define the differential operator

$$L = \frac{1 - i2^{-k}\langle \eta - \partial_{\tilde{x}}(\langle \tilde{\Phi}_\gamma(\tilde{x}, x)\tilde{x}, \xi\rangle), \partial_{\tilde{x}}\rangle}{1 + 2^{-k}|\eta - \partial_{\tilde{x}}\langle \tilde{\Phi}_\gamma(\tilde{x}, x)\tilde{x}, \xi\rangle|^2}$$

then after several more rounds of integration by parts, for any M and $\tilde{N}$ the last expression equals

$$\frac{(-i)^j}{(2\pi)^n} \iint e^{i(\langle \tilde{x}, \eta\rangle - \langle \tilde{\Phi}_\gamma(\tilde{x}, x)\tilde{x}, \xi\rangle)} \left(L^t\right)^M \left(\langle \nu', \mathrm{d}\Phi_\gamma^{-1}(x)\partial_\eta\rangle + \nu_r' \tilde{\Psi}_\gamma(\tilde{x}, x)_{pq}^r \partial_{\eta_p \eta_q}^2\right)^j \times \left(\frac{1 - 2^k \Delta_\eta}{1 + 2^k |\tilde{x}|^2}\right)^{\tilde{N}} \left(\tilde{\Phi}_\gamma(\tilde{x}, x)\partial_\eta\right)^\alpha \left(e^{i\langle x, \eta\rangle} \hat{f}(\eta)\right) \frac{\mathrm{d}\eta \, \mathrm{d}\tilde{x}}{|\det(\mathrm{d}\Phi_\gamma(\tilde{x}+x))|},$$

which may now be interpreted as an integral over $\mathbb{R}^{2n}$. Using definition 13 and the hypotheses on Φ_γ, the integrand in the previous formula can be bounded for any N by an expression of the form

$$C2^{-k(j+\frac{|\alpha|}{2})} \left(1 + 2^{-k/2}\|\eta - C_{\nu,k}\|\right)^{-N} \left(1 + 2^{-k/2}\left|\eta - \partial_{\tilde{x}}\left(\langle \tilde{\Phi}_\gamma(\tilde{x}, x)\tilde{x}, \xi\rangle\right)\right|\right)^{-M} \left(1 + 2^k |\tilde{x}|^2\right)^{-\tilde{N}}$$

for some positive C. Therefore, if $\tilde{N}$ and M are taken sufficiently large then

$$\begin{aligned} \left|\rho_{k'}^{1/2} \langle \nu', \partial_\xi \rangle^j \partial_\xi^\alpha \left(e^{i\langle \Phi_\gamma^{-1}(x), \xi\rangle} \widehat{\Phi_\gamma^* f}(\xi) \right)\right| &\lesssim 2^{-k(j+\frac{|\alpha|}{2})} \left(1 + 2^{-k/2}\left\|\xi - \mathrm{d}\Phi_\gamma^T(x) C_{\nu,k}\right\|\right)^{-N} \\ &\lesssim 2^{-k(j+\frac{|\alpha|}{2})} \left(1 + 2^{-k/2}\|\xi - C_{\nu',k'}\|\right)^{-N}. \end{aligned}$$

This is the required estimate and completes the proof. □

The next lemma says that we may decompose curvelet like functions into two pieces one of which is compactly supported in the frequency variable and the other which decays very quickly with the scale k.

Lemma 17. *If $\mathcal{F}$ is a family of curvelet like functions, then for every $\epsilon > 0$ it is possible to find a family of curvelet like functions $\mathcal{G}$ and a map, $T_{\mathcal{G}} : \mathcal{F} \to \mathcal{G}$ such that*

1. $\pi_\Gamma \circ T_{\mathcal{G}} = \pi_\Gamma$.
2. *For every* $(g,\gamma) \in \mathcal{G}$*, if* $\xi \in \operatorname{supp}(\hat{g})$*, then*

$$2^k(1/2-\epsilon) \leqslant |\xi| \leqslant 2^k(2+\epsilon) \quad \textit{and} \quad 2^k(1/2-\epsilon) \leqslant \left|\langle \nu, \xi\rangle\right| \leqslant 2^k(2+\epsilon).$$

3. *For every* $m \in \mathbb{R}$*,*

$$\left\{\left(2^{km}\left(f - \pi_{\mathcal{S}} \circ T_{\mathcal{G}}(f,\gamma)\right), \gamma\right)\right\}_{(f,\gamma)\in\mathcal{F}}$$

is a family of curvelet like functions.

Proof. We begin by choosing a cut-off function $\chi \in C_c^\infty$ supported within ϵ of the set $A_1 = \{2^{-1} \leqslant |\xi| \leqslant 2\} \cap \{2^{-1} \leqslant |\langle \nu, \xi\rangle| \leqslant 2\}$ and equal to 1 within $\epsilon/2$ of this set. We construct χ so that it is symmetric with respect to rotations that preserve ν. Also, we set $\chi_k(\xi) = \chi(2^{-k}\xi)$. The first task is to show that

$$\mathcal{G} = \left\{\chi_k(D)f, \gamma\right\}_{(f,\gamma)\in\mathcal{F}}$$

is a family of curvelet like functions which will then satisfy requirement 2. For $(f,\gamma) \in \mathcal{F}$ we have

$$\rho_k^{1/2}\langle \nu, \partial_\xi\rangle^j \partial_\xi^\alpha e^{i\langle x,\xi\rangle}\left[\widehat{\chi_k(D)f}\right](\xi) = \langle \nu, \partial_\xi\rangle^j \partial_\xi^\alpha\left(\rho_k^{1/2} e^{i\langle x,\xi\rangle}\chi_k(\xi)\hat{f}(\xi)\right)$$

and it follows from this expression and the Liebniz rule that $\mathcal{G}$ is a family of curvelet like functions.

It now remains to show that for any m,

$$\mathcal{H} = \left\{2^{km}\left(1-\chi_k(D)\right)f, \gamma\right\}_{(f,\gamma)\in\mathcal{F}}$$

is a family of curvelet like functions. Once again for $(f,\gamma) \in \mathcal{F}$, we have using definition 13 that for any N

$$\begin{aligned}
&\left|\rho_k^{1/2}\langle \nu, \partial_\xi\rangle^j \partial_\xi^\alpha e^{i\langle x,\xi\rangle}\left[\widehat{\left(1-\chi_k(D)\right)f}\right](\xi)\right| \\
&\quad = \left|\langle \nu, \partial_\xi\rangle^j \partial_\xi^\alpha\left(\rho_k^{1/2} e^{i\langle x,\xi\rangle}\left(1-\chi_k(\xi)\right)\hat{f}(\xi)\right)\right| \\
&\quad \lesssim 2^{-k(j+\frac{|\alpha|}{2})}\left(1+2^{-k/2}\|\xi - C_{\nu,k}\|\right)^{-N} \sup_{\|\xi - 2^k A_1\| > \epsilon 2^{k-1}} \left(1+2^{-k/2}\|\xi - C_{\nu,k}\|\right)^{-2m} \\
&\quad \lesssim 2^{-k(j+\frac{|\alpha|}{2})}\left(1+2^{-k/2}\|\xi - C_{\nu,k}\|\right)^{-N} 2^{-mk}.
\end{aligned}$$

This completes the proof. □

Now we begin to examine the action of pseudodifferential operators on families of curvelet like functions.

Lemma 18. *Suppose that* $\mathcal{F}$ *is a family of curvelet like functions, and that* $\{p_\gamma(y,\xi)\}_{\gamma\in\pi_\Gamma(\mathcal{F})}$ *is a collection of smooth functions on* $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ *such that for some* $m \in \mathbb{R}$*, any multi-indices* α *and* β*, and any nonnegative integer* j *there is a constant* $C_{\alpha,\beta,j}$ *so that*

$$\left|\partial_y^\beta \partial_\xi^\alpha \langle \nu, \partial_\xi\rangle^j p_\gamma(y,\xi)\right| \leqslant C_{\alpha,\beta,j} 2^{k\frac{|\beta|}{2}}\left(1+|\xi|\right)^{m-\frac{|\alpha|}{2}-j}$$

for all (y,ξ)*. Then*

$$\left\{\left(2^{-km} p_\gamma(y,D)f, \gamma\right)\right\}_{(f,\gamma)\in\mathcal{F}}$$

is a family of curvelet like functions.

Proof. We begin by choosing a family $\mathcal{G}$ as in Lemma 17 with some small value of $\epsilon > 0$. The following calculation then applies when (h,γ) equals either $T_{\mathcal{G}}(f,\gamma)$ or $(f - \pi_{\mathcal{S}} \circ T_{\mathcal{G}}(f,\gamma), \gamma)$,

$$\begin{aligned}
\rho_k^{1/2}\langle \nu, \partial_\xi\rangle^j \partial_\xi^\alpha e^{i\langle x,\xi\rangle}\mathscr{F}\left\{p_\gamma(y,D)h\right\}(\xi) &= \frac{1}{(2\pi)^n}\iint e^{i\langle \eta-\xi, y-x\rangle}\langle \nu, \partial_\eta\rangle^j \partial_\eta^\alpha p_\gamma(y,\eta)\left(\rho_k^{1/2} e^{i\langle x,\eta\rangle}\hat{h}(\eta)\right) \mathrm{d}\eta\, \mathrm{d}y \\
&= \frac{1}{(2\pi)^n}\iint e^{i\langle \eta-\xi, y\rangle}\langle \nu, \partial_\eta\rangle^j \partial_\eta^\alpha p_\gamma(y+x,\eta)\left(\rho_k^{1/2} e^{i\langle x,\eta\rangle}\hat{h}(\eta)\right) \mathrm{d}\eta\, \mathrm{d}y \\
&= \frac{1}{(2\pi)^n}\iint e^{i\langle \eta-\xi, y\rangle}\left(\frac{1-i2^{-k}\langle \eta-\xi, \partial_y\rangle}{1+2^{-k}|\eta-\xi|^2}\right)^M \left(\frac{1-2^k\Delta_\eta}{1+2^k|y|^2}\right)^N \\
&\quad \times \langle \nu, \partial_\eta\rangle^j \partial_\eta^\alpha p_\gamma(y+x,\eta)\left(\rho_k^{1/2} e^{i\langle x,\eta\rangle}\hat{h}(\eta)\right) \mathrm{d}\eta\, \mathrm{d}y.
\end{aligned}$$

In the case when $(h,\gamma)=T_{\mathcal{G}}(f,\gamma)$, by taking M sufficiently large this integral may be bounded by the required estimate since on the support of h

$$\left|\partial_y^\beta \partial_\eta^\alpha \langle \nu, \partial_\eta\rangle^j p_\gamma(y+x,\eta)\right| \lesssim 2^{k(m-j-\frac{|\alpha|}{2}+\frac{|\beta|}{2})}$$

and

$$\left(1+2^{-k/2}|\eta-\xi|\right)^{-1}\left(1+2^{-k/2}\|\eta-C_{\nu,k}\|\right)^{-1} \leqslant \left(1+2^{-k/2}\|\xi-C_{\nu,k}\|\right)^{-1}.$$

For the case when $(h,\gamma)=(f-\pi_{\mathcal{S}}\circ T_{\mathcal{G}}(f,\gamma),\gamma)$, we use the fact that $2^{k\tilde{m}}(f-\pi_{\mathcal{S}}\circ T_{\mathcal{G}}(f,\gamma))$ gives a family of curvelet like functions for any $\tilde{m}$. Therefore the integral in this case may be bounded by a constant times

$$2^{-k\tilde{m}}\left(1+2^{-k/2}\|\xi-C_{\nu,k}\|\right)^{-N}$$

for any $\tilde{m}$. This proves the result. □

The next lemma examines the case of Lemma 16 when the diffeomorphisms depend on a parameter.

Lemma 19. *Suppose that $\mathcal{F}$ is a family of curvelet like functions, and that $\{\Phi_\gamma\}_{\gamma\in\pi_\Gamma(\mathcal{F})}$ is a smooth family of functions from $[z_0,Z]\times\mathbb{R}^n$ to $\mathbb{R}^n$ such that*

$$\left\{\Phi_\gamma(z,\cdot)\right\}_{\gamma\in\pi_\gamma(\mathcal{F})}$$

satisfies the hypotheses of Lemma 16 for every fixed z with the constants in the estimates uniform with respect to z, and

$$\left|\partial_y^\alpha \partial_z \Phi_\gamma(z,y)\right| \leqslant C_\alpha 2^{k\frac{|\alpha|}{2}}.$$

Then

$$\left\{\left(2^{-k}\partial_z\left[\Phi_\gamma^*(z,\cdot)f\right],\Phi_\gamma^*(\gamma)\right)\right\}_{(f,\gamma)\in\mathcal{F}}$$

is a family of curvelet like functions.

Proof. Let $(f,\gamma)\in\mathcal{F}$. Then, as in the proof of Lemma 16 we use the Fourier inversion formula to establish that

$$\begin{aligned}\partial_z\left[\Phi_\gamma^*(z,\cdot)f\right](y) &= \frac{i}{(2\pi)^n}\iint e^{i\langle\Phi_\gamma(z,y),\xi\rangle}\left\langle\partial_z\Phi_\gamma(z,y),\xi\right\rangle\hat{f}(\xi)\,\mathrm{d}\xi\\ &= \Phi_\gamma^*(z,\cdot)\left[\left\langle\partial_z\Phi_\gamma\left(z,\Phi_\gamma^{-1}(z,\cdot)\right),\partial_y\right\rangle f\right](y).\end{aligned}$$

The collection of functions $\{\langle\partial_z\Phi_\gamma(z,\Phi_\gamma^{-1}(z,y)),\xi\rangle\}_{\gamma\in\pi_\gamma(\mathcal{F})}$ satisfy the hypotheses of Lemma 18 with $m=1$, and so that lemma and Lemma 16 imply the result. □

The next lemma gives an explicit expression for the leading order terms of the action of a suitable family of pseudo-differential operators with principal symbols that are homogeneous of degree 1 on a family of curvelet like functions. For every $\nu\in\pi_\nu(\mathcal{F})$ we use the notation P_ν for the matrix which gives orthogonal projection onto the space perpendicular to ν.

Lemma 20. *Suppose that $\mathcal{F}$ is a family of curvelet like functions, and that $\{p_\gamma\}_{\gamma\in\pi_\Gamma(\mathcal{F})}$ is a collection of smooth functions on $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$ such that*

$$\left|\partial_y^\beta\partial_\xi^\alpha p_\gamma(y,\xi)\right| \leqslant C_{\alpha,\beta}2^{k\frac{|\beta|}{2}}\left(1+|\xi|\right)^{1-|\alpha|},$$

and every p_γ is positive homogeneous in ξ of degree 1 on $\{2^{C-2}\leqslant|\langle\xi,\nu\rangle|\}$ where C is the constant from part 2 of Definition 13. Also, let $\phi(t)\in C_c^\infty(\mathbb{R})$ be a function that is equal to zero when $|t|\leqslant 2^{C-3}$ and equal to 1 when $|t|>2^{C-2}$. If for every $(f,\gamma)\in\mathcal{F}$ we define

$$g=2^{k/2}\left(p_\gamma(y,D)f-\left\langle\partial_\xi p_\gamma(y,\nu),Df\right\rangle-\frac{1}{2}\operatorname{tr}\left(\partial_\xi^2 p_\gamma(x,\nu)\frac{\phi(\langle\nu,D\rangle)}{\langle\nu,D\rangle}D^2 f\right)\right)$$

then

$$\left\{(g,\gamma)\right\}_{(f,\gamma)\in\mathcal{F}}$$

is a family of curvelet like functions.

Proof. First, applying both Lemma 17 and Lemma 18 we may assume without loss of generality that every $(f,\gamma)\in\mathcal{F}$ satisfies part 2 of Lemma 17 for some small value of $\epsilon>0$.

Next we make the following expansion of p_γ, which holds for $\xi\in\operatorname{supp}(\hat f)$ and follows using the homogeneity assumption:

$$p_\gamma(y,\xi)=\langle\partial_\xi p_\gamma(y,\nu),\xi\rangle+\frac{1}{2}\sum_{q,r=1}^{n}\partial_\xi^2 p_\gamma(x,\nu)_{qr}\frac{\xi^q\xi^r}{\langle\xi,\nu\rangle}+R_\gamma(y,\xi),\tag{A.4}$$

where

$$R_\gamma(y,\xi)=\frac{1}{2}\sum_{q,r,s=1}^{n}\left(\int_0^1(1-t)^2\partial^3_{\xi^q\xi^r\xi^s}p_\gamma\big(y,\langle\xi,\nu\rangle\nu+tP_\nu\xi\big)\,\mathrm{d}t\right)(P_\nu\xi)^q(P_\nu\xi)^r(P_\nu\xi)^s$$
$$+\frac{1}{2}\sum_{q,r,s=1}^{n}\left(\int_0^1\partial^3_{y^q\xi^r\xi^s}p_\gamma\big(x+t(y-x),\nu\big)\,\mathrm{d}t\right)(x-y)^q\frac{(P_\nu\xi)^r(P_\nu\xi)^s}{\langle\xi,\nu\rangle}.$$

Note that in order to obtain this formula we use the fact that ν lies in the kernel of the Hessian $\partial_\xi^2 p_\gamma(y,\nu)$ due to the homogeneity assumption. We write $(R^1_\gamma)_{qrs}(y,\xi)$ and $(R^2_\gamma)_{rs}(y)$ respectively for the two arrays of functions inside parentheses given by the integrals in the preceding formula. From the hypotheses and the fact that every $(f,\gamma)\in\mathcal{F}$ satisfies part 2 of Lemma 17 we see that each of these functions satisfy the hypotheses of Lemma 18 with respectively $m=-2$ and $m=0$ for ξ restricted to $\operatorname{supp}(\hat f)$ (in fact, $(R^2_\gamma)_{rs}$ satisfies the hypotheses with $m=-1/2$ when restricted in this way). Therefore by Lemma 15

$$\left\{2^{k/2}\left[R_\gamma(y,D)f\right],\gamma\right\}_{(f,\gamma)\in\mathcal{F}}$$

is a family of curvelet like functions. From (A.4) we observe that

$$g=2^{k/2}\left[R_\gamma(y,D)f\right],$$

and so the proof is complete. □

Appendix B. Lemma for paradifferential estimates

In this appendix we will state and prove the lemma used to deal with the "rough" parts of the Volterra kernels. The lemma is an extension of Lemma 13 in [25] to a broader class than just multipliers. The expansion methods used here can be found for example in [26] where credit is given to [27] for originating the ideas. See also [1, Theorem 4.5].

Let m a nonnegative integer, $a(x,\xi)\in C^{m,1}S^1_{cl}(\mathbb{R}^n)$ be homogeneous of order 1 in ξ, and a_k be obtained by (9) applied to a instead of p. Also, let $\beta\in C_0^\infty(\mathbb{R}^n)$ be a function such that $0\leqslant\beta\leqslant 1$, $\operatorname{supp}(\beta)\subset\{1/2\leqslant|\xi|\leqslant 2\}$ for some $l_0\in\mathbb{Z}^+$, and constructed so that $\beta_0(\xi)+\sum_{k=1}^\infty\beta(2^{-k+1}\xi)=1$ for another function $\beta_0\in C_0^\infty(\mathbb{R}^n)$ with support contained in the unit ball. For convenience we define $\beta_k(\xi)=\beta(2^{-k+1}\xi)$ for $k\geqslant 1$ (i.e. so that $\{\beta_k\}$ provides a Littlewood–Paley partition of unity). Assume that F_k is a family of operators on $L^2(\mathbb{R}^n)$ satisfying estimates of the form

$$\|F_k u\|_{L^2(\mathbb{R}^n)}\leqslant C2^{kr}\left\|\beta_k(D)u\right\|_{L^2(\mathbb{R}^n)}.\tag{B.1}$$

We will further assume that each F_k is frequency localized at the scale $A2^k$ for some constant A in the sense that $(1-\beta_k(D/A))F_k=R_k$ where $R_k:H^{s'}\to H^s$ is continuous uniform in k for all s' and s.

Lemma 21. *If $m\geqslant 0$ and $-(m+1)/2\leqslant s\leqslant m+1$ then there is an $N\in\mathbb{N}$ such that for any $u\in H^{s+r-(m-1)/2}(\mathbb{R}^n)$*

$$\left\|\left(\sum_k\big(a(y,D)-a_k(y,D)\big)F_k\right)u\right\|_{H^s}\lesssim\left(C+C'\right)\sum_{|\alpha|\leqslant 2N}\sup_{\omega\in\mathbb{S}^{n-1}}\left\|\partial_\xi^\alpha a(\cdot,\omega)\right\|_{C^{m,1}}\|u\|_{H^{s+r-\frac{m-1}{2}}}.$$

The constant C in this estimate is the same as the constant in (B.1), and C' is a uniform modulus of continuity for R_k acting between appropriate spaces.

Proof. For ease of notation we will write $f_k(y,\xi)=a(y,\xi)-a_k(y,\xi)$ and now record a few properties of f_k. First, from the homogeneity of a it is still true that f_k is homogeneous of degree 1 in ξ. Second, because a_k is obtained by a low pass filter in y from a, $\hat f_k(\eta,\xi)=\hat f_j(\eta,\xi)$ for $|\eta|\geqslant 2^{\max(k,j)/2}$. Finally, the estimates

$$\left|\partial_y^\beta\partial_\xi^\alpha f_k(y,\xi)\right|\lesssim 2^{-k(m+1-|\beta|)/2}\left\|\partial_\xi^\alpha a(\cdot,\xi)\right\|_{C^{m,1}}\tag{B.2}$$

for $|\beta|\leqslant m+1$ follow from (10).

The first step of the proof will be to decompose $f_k(y,D)$ in terms of a sum of multiplication and convolution operators by using spherical harmonics in the phase space. Indeed, let $\{w_\kappa\}$ denote the set of eigenfunctions of $\Delta_{\mathbb{S}^{n-1}}$, with the eigenvalue of w_κ denoted by λ_κ, which form an orthonormal basis for $L^2(\mathbb{S}^{n-1})$. By the homogeneity of f_k in ξ we have

$$f_k(y,\xi) = |\xi| \sum_\kappa f_{k\kappa}(y) w_\kappa\big(\xi/|\xi|\big),$$

where

$$f_{k\kappa}(y) = \int\limits_{\mathbb{S}^{n-1}} f_k(y,\omega) w_\kappa(\omega)\,\mathrm{d}\omega.$$

By Green's formula we have for any $N \in \mathbb{N}$

$$\lambda_\kappa^N f_{k\kappa} = \int\limits_{\mathbb{S}^{n-1}} f_k(y,\omega)\Delta^N_{\mathbb{S}^{n-1}} w_\kappa(\omega)\,\mathrm{d}\omega = \int\limits_{\mathbb{S}^{n-1}} \big(\Delta^N_{\mathbb{S}^{n-1}} f_k\big)(y,\omega) w_\kappa(\omega)\,\mathrm{d}\omega,$$

which using (B.2) gives

$$\big|\partial_y^\beta f_{k\kappa}\big| \lesssim \lambda_\kappa^{-N} 2^{-k(m+1-|\beta|)/2} \sum_{|\alpha|\leqslant 2N} \sup_{\omega\in\mathbb{S}^{n-1}} \big\|\partial_\xi^\alpha a(\cdot,\omega)\big\|_{C^{m,1}} \tag{B.3}$$

for $|\beta| \leqslant m+1$ and any N. Also, we can see from the definition that the family $\{f_{k\kappa}\}$ inherits the property that $\hat{f}_{k\kappa}(\eta) = \hat{f}_{j\kappa}(\eta)$ for $|\eta| \geqslant 2^{\max(k,j)/2}$ from $\{f_k\}$. With this decomposition we have

$$\sum_k f_k(y,D)F_k = \sum_{k,\kappa} f_{k\kappa}(y)|D| w_\kappa\big(D/|D|\big)F_k.$$

Following [1] we now split this sum to be estimated into two sums with even and odd k so that the sets where the F_k in each sum are concentrated do not overlap. Each of these sums may now be treated in the same way and so we focus only on the sum with even k which may be decomposed in the following way

$$\sum_{k\,\mathrm{even}} f_k(y,\xi)F_k u = \sum_{\kappa,k\,\mathrm{even}} f_{k\kappa}(y)\beta_k(D/A)v_\kappa - \sum_{\kappa,k\,\mathrm{even}} f_{k\kappa}(y)\beta_k(D/A) \sum_{j\,\mathrm{even},\,j\neq k} R_{j\kappa}u + \sum_{\kappa,k\,\mathrm{even}} f_{k\kappa}(y)R_{k\kappa}u, \tag{B.4}$$

where $v_\kappa = \sum_{j\,\mathrm{even}} |D| w_\kappa(D/|D|)F_j u$ and $R_{k\kappa} = |D| w_\kappa(D/|D|)R_k$. Note that using (B.1) and the estimate $\|w_\kappa\|_{L^\infty} \lesssim \lambda_\kappa^{\frac{n-1}{4}}$ we have

$$\|v_\kappa\|_{H^{s-\frac{m+1}{2}}} \lesssim C\lambda_\kappa^{\frac{n-1}{4}} \|u\|_{H^{s+r-\frac{m-1}{2}}}.$$

Since $R_{k\kappa}$ has the same mapping properties as R_k, but with norms bounded by $\lambda_\kappa^{\frac{n-1}{4}}$, if we take N to be large enough then the sums on the second line of (B.4) give an operator with the required properties provided that $|s| \leqslant m+1$ in which case $f_{k\kappa}$ acts as a multiplier mapping H^s to H^s with norm bounded by (B.3). Thus, using the rapid decay of $f_{k\kappa}$ in κ, we have reduced the proof to showing that operators of the form

$$\sum_k f_{k\kappa}(y)\beta_k(D/A)$$

map $H^{s-\frac{m+1}{2}}$ to H^s continuously with appropriately bounded norm for $-(m+1)/2 \leqslant s \leqslant m+1$. In fact this is already done for the case $m=1$ in [1].

We finally introduce one more decomposition so that the last sum becomes

$$S = \sum_{k,j} \beta_j(D/A) f_{k\kappa}(y)\beta_k(D/A). \tag{B.5}$$

Now we will look in more detail at the operators in this sum, which we will label as

$$\Gamma_{jk\kappa} = \beta_j(D/A) f_{k\kappa}\beta_k(D/A).$$

We will use the notation $l(A) = \min(0, \mathrm{floor}(\log_2(A)))$ and

$$\|a\|_N = \sum_{|\alpha|\leqslant 2N} \sup_{\omega\in\mathbb{S}^{n-1}} \big\|\partial_\xi^\alpha a(\cdot,\omega)\big\|_{C^{m,1}}.$$

Taking advantage of the frequency localization of $\beta(D/A)$ we may show

$$\|\Gamma_{jk\kappa}\|_{L^2\to L^2} \lesssim \begin{cases} A\lambda_\kappa^{-N}2^{-j(m+1)}\|a\|_N & j > k+3-l(A), \\ A\lambda_\kappa^{-N}2^{-k(m+1)/2}\|a\|_N & k+3-l(A) \geqslant j \geqslant k-3+l(A), \\ A\lambda_\kappa^{-N}2^{-k(m+1)}\|a\|_N & k-3+l(A) > j. \end{cases}$$

With these estimates available we now return to (B.5). For $v \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} \|Sv\|_{H^s}^2 &\lesssim C^2 \sum_{k,j} 2^{2js}\|\Gamma_{jk\kappa}\|_{L^2\to L^2}^2 \|\beta_k(D)v\|_{L^2}^2 \\ &\lesssim C^2 \sum_{k,j} 2^{2(js-k(s-\frac{m+1}{2}))}\|\Gamma_{jk\kappa}\|_{L^2\to L^2}^2 \big(2^{k(s-\frac{m+1}{2})}\|\beta_k(D)v\|_{L^2}\big)^2 \\ &\lesssim C^2\|a\|_N^2\Bigg(\sum_{j>k+3-l(A)} 2^{2(j(s-m-1)-k(s-\frac{m+1}{2}))}\big(2^{k(s-\frac{m+1}{2})}\|\beta_k(D)v\|_{L^2}\big)^2 \\ &\quad + \sum_{k-3+l(A)\leqslant j\leqslant k+3-l(A)} 2^{2(js-ks)}\big(2^{k(s-\frac{m+1}{2})}\|\beta_k(D)v\|_{L^2}\big)^2 \\ &\quad + \sum_{k>j+3-l(A)} 2^{2(js-k(s+\frac{m+1}{2}))}\big(2^{k(s-\frac{m+1}{2})}\|\beta_k(D)v\|_{L^2}\big)^2\Bigg). \end{aligned}$$

If $s < m+1$, then the first sum in parentheses is bounded by $\|u\|^2_{H^{s-\frac{m+1}{2}}}$. The second sum converges for any s and is also bounded by a constant times $\|u\|^2_{H^{s-\frac{m+1}{2}}}$. Finally, if $s > (m+1)/2$, then the third sum is bounded by the same quantity. This completes the proof of the lemma for $-(m+1)/2 < s < m+1$.

To complete the proof for the endpoints $s = -(m+1)/2$ and $s = m+1$ we first consider the case $m \geqslant 1$ where we use induction on m, and the fact mentioned above that the $m = 1$ case is already proven in [1]. Indeed, we already have that $S\partial_x : H^{\frac{m+1}{2}} \to H^m$ and $\partial_x S : H^{-\frac{m-1}{2}} \to H^0$, and so it suffices to show that

$$[\partial_x, S] : H^{s-\frac{m-1}{2}} \to H^s$$

for $s = m$ and $s = 0$. However,

$$[\partial_x, S] = \sum_k \partial_y f_{k\kappa}(y)\beta_k(D/A),$$

and the functions $\partial_y f_{k\kappa}$ have the same properties as $f_{k\kappa}$ but with m replaced by $m-1$. Using induction the proof is now complete for $m \geqslant 1$. In case $m = 0$, we use by the above that

$$S\langle D\rangle^{\frac12} : H^{\frac12} \to H^{\frac12}, \qquad \langle D\rangle^{\frac12} S : H^0 \to H^0.$$

Thus, it suffices to show that $[\langle D\rangle^{\frac12}, S] : H^s \to H^s$ for $s = \frac12$ and $s = 0$. In fact,

$$\big[\langle D\rangle^{\frac12}, S\big] : H^0 \to H^{\frac12},$$

as can be seen by interpolating the estimates in Propositions 4.1.B and 4.1.E of [26], since $S \in LipS^1_{cl}$. We remark that the estimates in [26] are stated for $A \in C^1S^1_{cl}$ but in fact hold for $A \in LipS^1_{cl}$, as seen by the Propositions 4.1.A and 4.1.D from which they are deduced. □

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