Adjoint state-based earthquake location

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Outline

1. Ray-based earthquake location

2. Adjoint state-based earthquake location
Ray-based earthquake location

\[ \delta x_s \rightarrow \delta t \quad \text{fixed } c(x) \]
Ray-based earthquake location

\[
\delta t = \int K_m(x) \delta \ln m(x) \delta(x - x_s) \, dl
\]

\[m(x) = (t, x_s)\]
Ray-based earthquake location

\[ K_{t_0}(x) = -1 \]

\[ K_{\theta_s}(x) = \frac{(R - h) \sin i \cos \alpha \theta_s}{V_x} \]

\[ K_{\phi_s}(x) = -\frac{(R - h) \sin i \sin \theta \sin \alpha \varphi_s}{V_x} \]

\[ K_{z_s}(x) = -\frac{\cos i z_s}{V_x} \]
1) **The relationship between \( \delta t \) and \( \delta u \)**

The cross-correlation between synthetic data and observed data is:

\[
C(\tau) = \int_{t_1}^{t_2} u(t - \tau) u_{obs}(t) dt = \int_{t_1}^{t_2} u(t - \tau) [u(t) + \delta u(t)] dt
\]

Do Taylor expansion for \( u(t - \tau) = u(t) - \tau \frac{\partial u(t)}{\partial t} + \frac{1}{2} \tau^2 \frac{\partial^2 u(t)}{\partial t^2} + O(\tau^3) \).

We have

\[
C(\tau) = \int_{t_1}^{t_2} \left[ u(t) - \tau \frac{\partial u(t)}{\partial t} + \frac{1}{2} \tau^2 \frac{\partial^2 u(t)}{\partial t^2} \right] u(t) dt + \int_{t_1}^{t_2} \left[ u(t) - \tau \frac{\partial u(t)}{\partial t} + \frac{1}{2} \tau^2 \frac{\partial^2 u(t)}{\partial t^2} \right] \delta u(t) dt
\]
1) The relationship between $\delta t$ and $\delta u$

$$\partial_\tau C(\tau) = \int_{t_1}^{t_2} \left[ -\frac{\partial u(t)}{\partial t} + \tau \frac{\partial^2 u(t)}{\partial t^2} \right] u(t) dt + \int_{t_1}^{t_2} -\frac{\partial u(t)}{\partial t} \delta u(t) dt = 0$$

$$\int_{t_1}^{t_2} -\frac{\partial u(t)}{\partial t} u(t) dt = u^2(t_2) - u^2(t_1) = 0 = u(t_1) = u(t_2)$$

So, we get the the time shift due to the displacement perturbation:

$$\delta t = \tau = \frac{\int_{t_1}^{t_2} \dot{u}(t) \delta u(t) dt}{\int_{t_1}^{t_2} \ddot{u}(t) u(t) dt} = \frac{\langle \dot{u}(t), \delta u(t) \rangle_t}{\langle \ddot{u}(t), u(t) \rangle_t}$$

The above mathematical derivation does not involve in wave equation theory.
2) The relationship between $\delta u$ and $\delta f$

$$\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot [T + \delta T] = f$$

The solution:

$$u = G \otimes f$$

$$u + \delta u = (G + \delta G) \otimes (f + \delta f)$$

We get the difference:

$$\delta u = G \otimes \delta f + \delta G \otimes f \approx G \otimes \delta f$$
3) **Calculate the corresponding kernel of source**

\[
\delta \chi = \sum_{r=1}^{N_r} [t_{sr} - t_{sr}^{obs}] \delta t_{sr}
\]

*Using the following relationship*

\[
\delta t = \frac{\langle \dot{u}(t), \delta u(t) \rangle_t}{\langle \ddot{u}(t), u(t) \rangle_t}
\]

\[
\delta u = G \otimes \delta f
\]

So,

\[
\delta \chi = \sum_{r=1}^{N_r} [t_{sr} - t_{sr}^{obs}] \frac{\langle \dot{u}(t), \delta u(t) \rangle_t}{\langle \ddot{u}(t), u(t) \rangle_t}
\]

\[
= \sum_{r=1}^{N_r} [t_{sr} - t_{sr}^{obs}] \frac{\langle \dot{u}(t), G \otimes \delta f \rangle_t}{\langle \ddot{u}(t), u(t) \rangle_t}
\]
3) **Calculate the corresponding kernel of source**

$$\delta \chi = \sum_{r=1}^{N_r} [t_{sr} - t_{sr}^{obs}] \frac{<\ddot{u}(t), G \otimes \delta f>_t}{<\ddot{u}(t), u(t)>_t}$$

*Since we have:*

$$<h, G \otimes f>_t = <G^* \otimes h, f>_t$$

$$G^*(t) = G(-t)$$

*we further get:*

$$\delta \chi = \sum_{r=1}^{N_r} [t_{sr} - t_{sr}^{obs}] \frac{<G^* \otimes u(t), \delta f>_t}{<\ddot{u}(t), u(t)>_t}$$
3) Calculate the corresponding kernel of source

\[
\delta \chi = \sum_{r=1}^{N_r} [t_{sr} - t_{sr}^{obs}] \frac{\langle G^* \otimes \dot{u}(t), \delta f \rangle_t}{\langle \dot{u}(t), u(t) \rangle_t}
\]

The adjoint source:

\[
f^+ = \sum_{r=1}^{N_r} [t_{sr} - t_{sr}^{obs}] \frac{\dot{u}(t)}{\langle \dot{u}(t), u(t) \rangle_t}
\]

The adjoint wavefield:

\[
s^+ = G(-t) \otimes f^+
\]

So the variation of cost function:

\[
\delta \chi = \langle s^+, \delta f \rangle_t
\]
3) **Calculate the corresponding kernel of source**

\[ f(x, y, t) = h(t) \ast \delta(x - x_s) \ast \delta(y - y_s) \]

\[ h(t) = \frac{-2\alpha^3}{\sqrt{\pi}(t - t_s)}\exp(-\alpha^2(t - t_s)^2) \]

*Taking the variation of \( f(x, y, t) \), we get*

\[ \delta f(x, y, t) = \dot{h}(t) \ast \delta(x - x_s) \ast \delta(y - y_s) \ast \delta t_s + \]

\[ h(t) \ast \partial_{x_s} [\delta(x - x_s) \ast \delta(y - y_s)] \ast \delta x_s + \]

\[ h(t) \ast \partial_{y_s} [\delta(y - y_s) \ast \delta(y - y_s)] \ast \delta y_s \]

\[ \delta \chi = < s^+, \delta f >_t = \int_0^T \int_\Omega \delta f(x, y, t)s^+(x, y, T - t)dx dy dt \]
3) Calculate the corresponding kernel of source

\[ \delta f(x, y, t) = -\dot{h}(t) \cdot \delta(x - x_s) \cdot \delta(y - y_s) \cdot \delta t_s + h(t) \cdot \partial_{x_s} [\delta(x - x_s) \cdot \delta(y - y_s)] \cdot \delta x_s + h(t) \cdot \partial_{y_s} [\delta(y - y_s) \cdot \delta(y - y_s)] \cdot \delta y_s \]

Then, recalling the cost function

\[ \delta \chi = \langle s^+, \delta f \rangle_t \]

\[ = \int_0^T \int_{\Omega} \delta f(x, y, t)s^+(x, y, T - t)dx dy dt \]

\[ = -\int_0^T \dot{h}(t) \cdot s^+(x_s, y_s, T - t) dt \cdot \delta t_s + \int_0^T h(t) \cdot \partial_{x_s} s^+(x_s, y_s, T - t) dt \cdot \delta x_s + \]

\[ \int_0^T h(t) \cdot \partial_{y_s} s^+(x_s, y_s, T - t) dt \cdot \delta y_s \]
3) Calculate the corresponding kernel of source

\[ \delta \chi = < s^+, \delta f >_t \]

\[ = - \int_0^T \dot{h}(t) * s^+(x_s, y_s, T - t) dt \delta t_s + \int_0^T h(t) * \partial_{x_s} s^+(x_s, y_s, T - t) dt \delta x_s + \]

\[ \int_0^T h(t) * \partial_{y_s} s^+(x_s, y_s, T - t) dt \delta y_s \]

\[ k_{t_s} = - \int_0^T \dot{h}(t) * s^+(x_s, y_s, T - t) dt \]

\[ k_{x_s} = \int_0^T h(t) * \partial_{x_s} s^+(x_s, y_s, T - t) dt \]

\[ k_{y_s} = \int_0^T h(t) * \partial_{y_s} s^+(x_s, y_s, T - t) dt \]
Thanks