Spiking Deconvolution for Seismic Waves Using the Fractional Fourier Transform

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***Abstract*—This paper applies the Fractional Fourier Transform (FrFT) to a collected seismic trace to estimate the Earth's reflectivity function. This is done by computing the FrFT domain 'a' in which the source wavelet is as close to a delta (or spiking) function as possible, mimicking the concept of spiking deconvolution. We show by simulation that the proposed method outperforms conventional spiking deconvolution (SD) and time domain deconvolution (TDD) by nearly an order of magnitude over signal-to-noise ratios (SNRs) of -10 to 20 dB.**

I. Introduction

Significant research in the field of geosciences has been in the subject area of seismological characterization of the earth's layers. This field is important for understanding the detailed structure of the earth, which may prove useful in future prediction of earthquakes. Currently, there are no reliable methods for predicting earthquakes, but they are directly correlated with faults and other characteristics of the earth's subterranean layers. Techniques used for predicting the earth's structure, including spiking deconvolution (SD) or time domain deconvolution (TDD), are therefore useful. In SD, a known or estimated source signal that is produced by some seismological event within the earth, propagates to the collector, and is correlated with the collected seismological signal [9] (an alternate solution is to estimate the inverse of the source transfer function and convolve it with the received seismic signal [4]). The output of the correlation provides a measure of the reflectivity at each boundary of the layers and formulates a picture of what those boundaries look like. In TDD, deconvolution of the received signal with the source wavelet provides another way of measuring the reflectivity function [9]. Both methods are limited in accuracy, because the source wavelet interferes with the measurement. Since the source wavelet is not an impulse, measuring the reflectivity, or impulse response of the earth, is harder.

In this paper, we propose to perform deconvolution using the Fractional Fourier Transform (FrFT). The proposed approach computes the fractional domain 'a' in which the source wavelet is compressed into as close to a single delta function (or spike) as possible. Then, we perform deconvolution with the delta function and the received seismic signal, translated to the same domain, to extract the Earth's reflectivity in the FrFT domain, finally rotating back to the time domain to obtain the earth's response. We show by simulation that this method gives nearly an order of magnitude improvement in estimating the reflectivity function of the earth using mean-square error (MSE) between the true and estimated reflectivity as the performance metric.

The paper outline is as follows: Section II presents the signal model and deconvolution problem. Section III discusses conventional spiking deconvolution (SD) and time domain deconvolution (TDD). Section IV presents the proposed Fractional Fourier domain deconvolution (FrFT-DD) approach, which is the FrFT version of TDD. Section V presents simulation results comparing the SD, TDD and FrFT-DD methods, showing the advantage of the FrFT-DD. Conclusions and remarks on future work are given in Section VI.

II. Signal Model

When a source (seismic) signal, e.g. due to an earthquake, is produced, it propagates to the surface of the earth. It is attenuated and delayed as it hits each layer, and this can be modeled by a series of impulse functions known as the earth's reflectivity function. The result is the source signal is convolved with this reflectivity function. We assume that the received seismic signal takes the discrete time form

$$s(i) = w(i) * r_e(i) + n(i), \tag{1}$$

where $w(i)$ is the source seismic signal, i.e. the propagating signal generated from some seismic event within the earth's layers, usually taking on the form of a wavelet. Furthermore, $r_e(i)$ is the reflectivity function of the earth, modeled as a sum of impulse functions, and '*' denotes convolution; $n(i)$ is additive white Gaussian noise (AWGN), and $s(i)$ is the received seismic signal collected at the earth's surface [4].

In certain cases, the problem is to determine $w(i)$ assuming that $r_e(i)$ is known or can be estimated. An example of this is when the seismic trace is measured near a borehole, which is a long, narrow hole made in the ground, often used to determine soil conditions or locate substances such as oil, water, or minerals. In this paper, we assume we know or can estimate the source signal $w(i)$. The problem then, is to determine the reflectivity function $r_e(i)$ from the received

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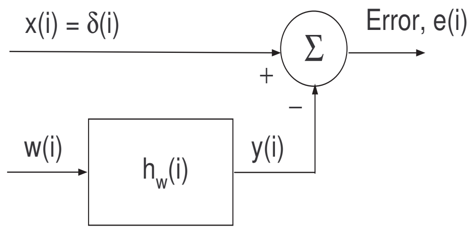

Fig. 1. Spiking Deconvolution as Wiener Filtering Problem

signal $s(i)$, also called a seismic trace [4]. Note that typically multiple traces will be collected and processed to improve the estimation accuracy.

III. Conventional Methods

A. Spiking Deconvolution (SD)

In spiking deconvolution (SD), we estimate the inverse of the source wavelet $w(i)$, denoted $h_w(i)$, such that [4]

$$w(i) * h_w(i) = \delta(i), \tag{2}$$

where $\delta(i)$ is the Dirac delta function defined by [5]

$$\delta(i) = \{ \begin{array}{c} 1, i = 0 \\ 0, i \neq 0 \end{array} . \tag{3}$$

We can compute $h_w(i)$ using standard Wiener filtering concepts. Referring to Fig. 1 and Eq. (2), the filter coefficients $h_w(i)$ are chosen to minimize the mean-square error (MSE) between $y(i)$, which is the filter output, and $x(i)$, the desired output of the convolution between $w(i)$ and $h_w(i)$, a delta function. The goal is to minimize the error, $e(i)$, between $y(i) = w(i) * h_w(i)$ and $\delta(i)$. Assuming we compute an average over $i = 1, 2, ..., N$ samples,

$$e = \frac{1}{N}\sum_{i=1}^{N}|\delta(i) - y(i)|^2 = \frac{1}{N}\sum_{i=1}^{N}|\delta(i) - w(i) * h_w(i)|^2. \tag{4}$$

This least-squares (LS) problem is solved by computing the filter coefficients h_w such that

$$\partial e / \partial h_w = 0. \tag{5}$$

The solution is [4]

$$\mathbf{h}_w = \mathbf{R}^{-1}\mathbf{r}_{w\delta}, \tag{6}$$

where $\mathbf{R} = \mathbf{w}\mathbf{w}^H$ is the covariance matrix of the input source wavelet and $\mathbf{r}_{w\delta} = \mathbf{w}\delta$ is the cross-correlation between the source wavelet and the delta function, and boldface notation denotes the vector (capital letters for a matrix) produced by an ensemble average over N samples. Once we obtain $\mathbf{h}_w$, the estimate of the earth's reflectivity function is given by

$$\hat{\mathbf{r}}_e = deconv(\mathbf{s}, \mathbf{h}_w). \tag{7}$$

This algorithm does not perform well in practice, because inversion of the sample covariance matrix typically requires more samples than are available [7], instabilities in deconvolution can occur [9], and deconvolution is an imperfect operator [9]. Hence, we propose a better approach in Section IV.

Note that another method for performing spiking deconvolution, as described in [9], is to cross-correlate the received seismic signal with the source wavelet. That is, we compute $\hat{\mathbf{r}}_e = xcorr(\mathbf{s}, \mathbf{w})$. This results in a pattern that closely resembles a series of impulses representing the reflectivity. However, when the impulses are closely spaced, this method does not work well. Hence, we do not consider it further.

B. Time Domain Deconvolution (TDD)

Time domain deconvolution is a simpler solution where we use our known or estimated version of $\mathbf{w}$ to directly compute

$$\hat{\mathbf{r}}_e = deconv(\mathbf{s}, \mathbf{w}). \tag{8}$$

Note that inaccuracies in the estimate will result because the source wavelet $\mathbf{w}$ is not a delta function and interferes with the estimate of the reflectivity function $\mathbf{r}_e$ [9], and we do not transform it into a delta function as with spiking deconvolution. Deconvolution and AWGN will also cause errors in the estimate [9].

IV. Proposed FrFT Domain Deconvolution

In this section, we attempt to improve estimation of the reflectivity function $\mathbf{r}_e$ using the Fractional Fourier Transform (FrFT). The continuous time FrFT and its properties, as well as its relationship to the Wigner Distribution (WD), are well-understood (e.g. [2], [3], and [6]). In discrete time, we model the $N \times 1$ FrFT of an $N \times 1$ vector $\mathbf{x}$ by writing

$$\mathbf{X}_a = \mathbf{F}^a\mathbf{x}, \tag{9}$$

where $0 < |a| < 2$, $\mathbf{F}^a$ is an $N \times N$ matrix whose elements are given by ([1] and [6])

$$\mathbf{F}^a[m, n] = \sum_{k=0, k \neq (N-1+(N)_2)}^{N} u_k[m] e^{-j\frac{\pi}{2}ka} u_k[n], \tag{10}$$

$u_k[m]$ and $u_k[n]$ are the eigenvectors of the matrix $\mathbf{S}$ [1]

$$\mathbf{S} = \begin{bmatrix} C_0 & 1 & 0 & \dots & 1 \\ 1 & C_1 & 1 & \dots & 0 \\ 0 & 1 & C_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & C_{N-1} \end{bmatrix}, \tag{11}$$

and

$$C_n = 2cos(\frac{2\pi}{N}n) - 4. \tag{12}$$

The key idea behind the proposed approach is to translate both the source wavelet $\mathbf{w}$ and the collected seismic signal s into a domain 'a' using the FrFT where the source wavelet is as close to a delta function as possible ([8] and [9]). When the wavelet approaches a delta function, all of the energy becomes concentrated at one sample, hence the amplitude of that sample increases in accordance with energy conservation. Hence, finding the best value of 'a' amounts to searching over the range $0 \leq a \leq 2$ and choosing the value for which the peak magnitude of the FrFT of the wavelet is maximum. The search requires a choice of step size, and we arbitrarily set this to $\Delta a = 0.01$. We then perform deconvolution as before, in the new domain, finally translating the result back to the time domain through an inverse FrFT (IFrFT), using the same value of 'a' as before, to obtain the earth's reflectivity estimate. This method closely mimics spiking deconvolution but does not require matrix inversions or lots of samples, hence it will perform more accurately in a non-stationary, real-world environment [6]. This method also improves performance over time domain deconvolution because now the source wavelet is at or near a delta function. The proposed algorithm is written mathematically in Table I, and the estimated reflectivity function is

$$\hat{\mathbf{r}}_e = \mathbf{F}^{-a_{opt}}\mathbf{R}_e(a_{opt}). \tag{13}$$

TABLE I
PROPOSED FRFT DOMAIN DECONVOLUTION (FRFT-DD) ALGORITHM

1. For $a = 0 : \Delta a : 2$ % Loop over all a
$\mathbf{W}(a) = \mathbf{F}^a\mathbf{w}$; % Compute FrFT of $\mathbf{w}$
$\mathbf{W}_{max}(a) = max(\lvert\mathbf{W}(a)\rvert)$; % Compute max value
End
2. Find peak over all a to get the optimum
$a_{opt} = \underset{a}{arg\ max}\ W_{max}(a)$;
3. Compute the FrFT, using a_{opt}, of the source wavelet and received seismic signal
$\mathbf{W}(a_{opt}) = \mathbf{F}^{a_{opt}}\mathbf{w}$;
$\mathbf{S}(a_{opt}) = \mathbf{F}^{a_{opt}}\mathbf{s}$;
4. Compute the estimated reflectivity function using deconvolution
$\mathbf{R}_e(a_{opt}) = deconv(\mathbf{S}(a_{opt}), \mathbf{W}(a_{opt}))$;
5. Rotate back to the time domain
$\hat{\mathbf{r}}_e = \mathbf{F}^{-a_{opt}}\mathbf{R}_e(a_{opt})$;

V. SIMULATIONS

The source wavelet we assume for the simulations is shown in Fig. 2, containing $N = 64$ samples. We run $M = 1000$ trials to compute an average of the MSE. We plot the MSE between the reflectivity estimates using the SD, TDD, and FrFT-DD algorithms, given in Eqs. (7), (8), and (13), respectively, and the true reflectivity function. The (root) MSE is given by

$$MSE = \sqrt{\sum_{i=1}^{N}(r_e(i) - \hat{r_e}(i))^2/N}. \tag{14}$$

The MSE is shown as a function of signal-to-noise ratio (SNR), which is varied by changing the amplitude of the AWGN, $n(i)$. We test four different reflectivity functions, using lengths of $L = 3, 7, 11$, and 15, with results shown in Figs. 3 - 6, respectively. Note that the impulses in $\mathbf{r}_e$ are

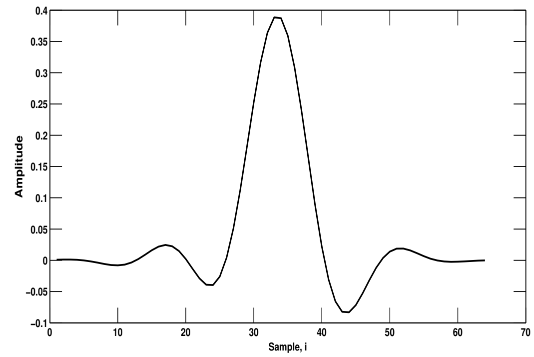

Fig. 2. Source Wavelet $\mathbf{w}$

closely spaced here; more widely spaced impulses will give better results. In all cases, the FrFT-DD algorithms provides nearly a tenfold improvement over all SNR, whereas the other two algorithms do not perform well. Initially, when $L = 3$, the SD algorithm does slightly better than the TDD algorithm, but it degrades quickly as L increases. The SD algorithm does not do as well due to the lack of sample support [7], since we have only $N = 64$ samples in the source wavelet, and $N + L - 1$ samples in the received seismic trace. The TDD method also does not perform as well because the source wavelet has not been reduced to a spike (delta) function, but it does not degrade as L increases, since it is not sample support dependent. The FrFT-DD does not significantly degrade as L increases either, showing its robustness as the reflectivity, and hence the structure of the earth's layers, changes. Slight variation in performance is expected because of the randomly generated reflectivity function. Note also that the FrFT-DD algorithm is noise dominated at low SNR but dominated by the source wavelet compression ability at high SNR, hence performance flattens out.

VI. CONCLUSION

In this paper, we present a new method based on the Fractional Fourier Transform (FrFT) to estimate the earth's reflectivity from a seismic trace by translating the known or estimated source wavelet and received signal to the FrFT domain in which the source wavelet most closely resembles a delta function (spike). This enables spiking deconvolution in the FrFT domain, which reduces error over traditional time domain methods because it does not require large sample support, and it reduces deconvolution inaccuracies by compressing the source wavelet to a spike. Improvements are nearly an order of magnitude when the impulses contained in the reflectivity function are closely spaced, as the new FrFT-DD algorithm is able to better separate the reflectivity from the source wavelet. Future work is in improving the algorithm presented to achieve further reduction in MSE and to apply the FrFT to other problems in the fields of geoscience or remote sensing.

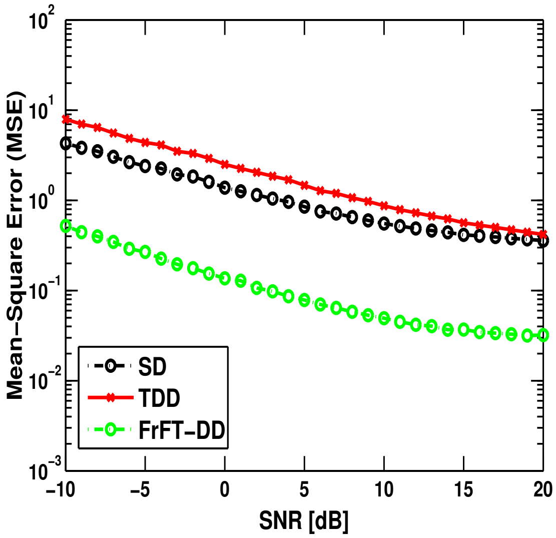

Fig. 3. $L = 3$ Length Reflectivity Function, $\mathbf{r}_e$

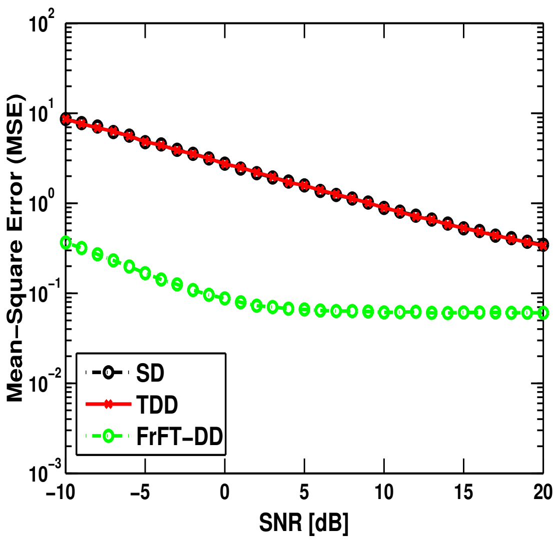

Fig. 4. $L = 7$ Length Reflectivity Function, $\mathbf{r}_e$

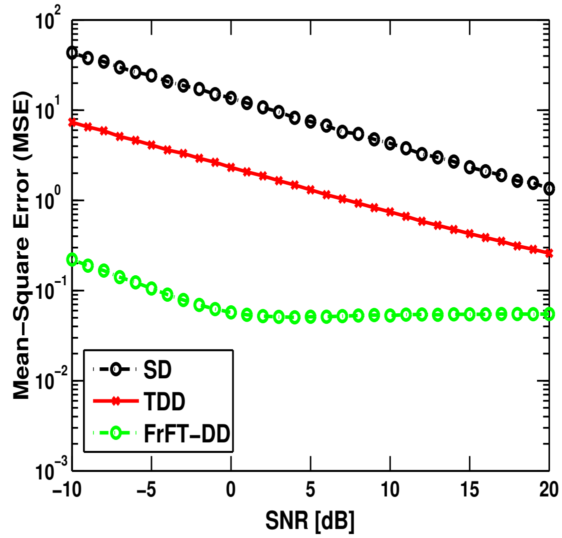

Fig. 5. $L = 11$ Length Reflectivity Function, $\mathbf{r}_e$

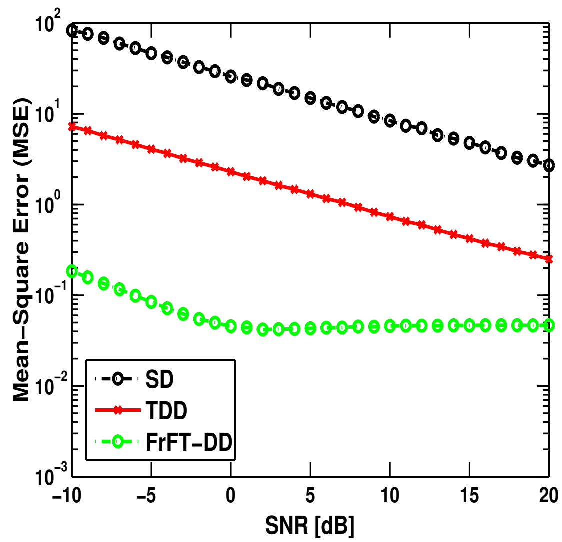

Fig. 6. $L = 15$ Length Reflectivity Function, $\mathbf{r}_e$

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