Minor-Closed Classes of Polymatroids

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These slides are available at
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Polymatroids

A (discrete or integer) **polymatroid** on a set $E$ is a function $\rho : 2^E \to \mathbb{Z}$ that is

- **normalized**: $\rho(\emptyset) = 0$,
- **non-decreasing**: $\rho(A) \leq \rho(B)$ for all $A \subseteq B \subseteq E$, and
- **submodular**: $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ for all $A, B \subseteq E$.

Polymatroids generalize matroids by allowing elements to be loops, points, lines, planes, ..., not just loops or points.

A 2-polymatroid: $\rho(\emptyset) = 0$; $\rho(d) = 1$; sets $X$ with $\rho(X) = 2$: $\{a\}$, $\{b\}$, $\{c\}$, $\{a, d\}$, $\{b, d\}$; the rest have $\rho(X) = 3$. 
A polymatroid $\rho$ on $E$ is a $k$-polymatroid if $\rho(e) \leq k$ for all $e \in E$.

Matroids are 1-polymatroids.

Minors are defined as for matroids, via $\rho$: for $A \subseteq E$,
- deletion: $\rho\backslash_A(X) = \rho(X)$ for $X \subseteq E - A$,
- contraction: $\rho/_{A}(X) = \rho(X \cup A) - \rho(A)$ for $X \subseteq E - A$,
- minors: any combination of deletion and contraction.

The class $\mathcal{P}_k$ of $k$-polymatroids is minor-closed, that is, all minors of polymatroids in $\mathcal{P}_k$ are in $\mathcal{P}_k$.

There is one excluded minor for $\mathcal{P}_k$ for each $t > k$: $\rho$ on $\{e\}$ with $\rho(e) = t$. 
The **Boolean polymatroid** of a graph $G = (V, E)$ is the 2-polymatroid $\rho_G$ on $E$ with $\rho_G(X) = |V(X)|$, where $V(X) = \{v \in V : v \text{ is incident with at least one edge in } X\}$.

**Proposition**

Apart from isolated vertices, we can reconstruct $G$ from $\rho_G$.

Note that $\rho_G(X) = 2|X|$ if and only if $X$ is a matching.

Boolean polymatroids have substantial implications for matching theory.
The excluded minors for Boolean polymatroids

The class of Boolean polymatroids, extended by allowing polymatroid loops, is minor-closed (deletion is graph deletion; contraction is a variant on graph deletion).

There are eight excluded minors (within the class of 2-polymatroids).

Oxley and Whittle
The $k$-dual $\rho^*$ of a $k$-polymatroid $\rho$ on $E$: for $X \subseteq E$, 

$$\rho^*(X) = k|X| - \rho(E) + \rho(E - X).$$

The $k$-dual: is a $k$-polymatroid; 
depends on $k$; 
is an involution: $(\rho^*)^* = \rho$; 
relates deletion and contraction:

$$(\rho \setminus A)^* = (\rho^*) / A \quad \text{and} \quad (\rho / A)^* = (\rho^*) \setminus A.$$
A way to get some $k$-polymatroids

For matroids $M_1, M_2, \ldots, M_k$ on $E$, defining $\rho : 2^E \to \mathbb{Z}$ by

$$\rho(X) = r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_k}(X),$$

for $X \subseteq E$, gives a $k$-polymatroid. We say $\rho$ is $k$-decomposable.
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\]

for \(X \subseteq E\), gives a \(k\)-polymatroid. We say \(\rho\) is \(k\)-decomposable.

Decompositions interact well with minors and the \(k\)-dual:

\begin{itemize}
  \item \(\rho \backslash A = r_{M_1 \backslash A} + r_{M_2 \backslash A} + \cdots + r_{M_k \backslash A}\),
  \item \(\rho / A = r_{M_1 / A} + r_{M_2 / A} + \cdots + r_{M_k / A}\), and
  \item \(\rho^* = r_{M_1^*} + r_{M_2^*} + \cdots + r_{M_k^*}\).
\end{itemize}
Not all polymatroids are decomposable

A counterpart, for 2-polymatroids, of the Vámos matroid:

\[
\rho(X) = \begin{cases} 
2|X|, & \text{if } |X| \leq 1, \\
3, & \text{if } |X| = 2 \text{ and } X \neq \{a, d\}, \\
4, & \text{otherwise.}
\end{cases}
\]

This 2-polymatroid is isomorphic to its 2-dual.
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A notion suggested by Boolean polymatroids:

An incidence set in a polymatroid \( \rho \) on \( E \) is a subset \( X \) of \( E \) with \( |X| \geq 2 \) and \( \rho(X') = |X'| + 1 \) for all \( X' \subseteq X \) with \( 1 \leq |X'| \leq 3 \).
Not all polymatroids are decomposable

Incidence set: \( |X| \geq 2 \) and \( \rho(X') = |X'| + 1 \) for all \( X' \subseteq X \) with \( 1 \leq |X'| \leq 3 \).

**Lemma**

<table>
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| Let \( \rho \) be \( r_{M_1} + r_{M_2} + \cdots + r_{M_k} \) for matroids \( M_1, M_2, \ldots, M_k \) on \( E \).
| 1. For any incidence set \( X \) of \( \rho \), there is exactly one \( M_i \) in which all elements of \( X \) are parallel. |
| 2. Let \( X \) and \( Y \) be incidence sets. If \( X \cap Y \neq \emptyset \) but \( \rho(\{a, b\}) = 4 \) for some \( a \in X \) and \( b \in Y \), then \( |X \cap Y| = 1 \). |

Using part (2) with \( \{a, b, c\} \) and \( \{b, c, d\} \) shows that \( \rho \) is not decomposable.

For all \( k \geq 2 \), \( \rho \) is an excluded minor for the minor-closed class \( \mathcal{D}_k \) of \( k \)-decomposable polymatroids.
Which polymatroids are of this type?

Which polymatroids are decomposable? (Murty and Simon, 1978.)

What are the excluded-minors for $D_k$?

Even $D_2$ seems to be open.

What are the excluded-minors for $\bigcup_{k \geq 2} D_k$?
Which polymatroids are of this type?

Which polymatroids are decomposable? (Murty and Simon, 1978.)

What are the excluded-minors for $\mathcal{D}_k$?

Even $\mathcal{D}_2$ seems to be open.

What are the excluded-minors for $\bigcup_{k \geq 2} \mathcal{D}_k$?

Given $\rho$, there may be many options for $M_1, M_2, \ldots, M_k$.

Lemos (2002) showed how, given one decomposition of a 2-polymatroid $\rho$, to get all decompositions of $\rho$. 


D. Chun (2009) studied deletion-contraction polymatroids, or dc-polymatroids, that is, 2-polymatroids of the form

$$\rho(X) = r_{M\setminus y}(X) + r_{M/y}(X)$$

for $X \subseteq E$, for some matroid $M$ on $E \cup y$.

(Vertigan; Geelen, Gerards, and Whittle.)

A 2-polymatroid that is not a dc-polymatroid:
A special case: dc-polymatroids

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A 2-polymatroid that is not a dc-polymatroid:

$$\rho$$

- $f$
- $e$

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Theorem

D. Chun, 2009

The excluded minors for the minor-closed class of dc-polymatroids, within $P_2$, are $\rho$ (above) and $U_{2,2}$. 
A matroid $Q$ is a quotient of $L$, or $L$ is a lift of $Q$, if $L = M \setminus A$ and $Q = M / A$ for some matroid $M$ and $A \subseteq E(M)$.

E.g., extend the uniform matroid $U_{5,9}$ on $\{1, 2, \ldots, 9\}$ by three elements using the modular cuts generated by

- $\{1, 2, 3\}$ and $\{4, 5, 6\}$,
- $\{1, 2, 3\}$ and $\{7, 8, 9\}$,
- $\{4, 5, 6\}$ and $\{7, 8, 9\}$,

and then contract the added elements to get the quotient

```
1, 2, 3   4, 5, 6   7, 8, 9
```
For matroids $Q$ and $L$ on $E$, the following are equivalent:

- $Q$ is a quotient of $L$;
- $L^*$ is a quotient of $Q^*$;
- $r_L - r_Q$ is non-decreasing; that is, for all $X \subseteq Y \subseteq E$,
  \[
  r_L(X) - r_Q(X) \leq r_L(Y) - r_Q(Y),
  \]
  or, equivalently,
  \[
  r_Q(X \cup e) - r_Q(X) \leq r_L(X \cup e) - r_L(X)
  \]
  for all $X$ and $e \in E - X$. 

Quotients
A broader special case

A polymatroid $\rho$ of the form $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ where each matroid $M_{i+1}$ is a quotient of $M_i$ is a $k$-quotient polymatroid.

The class $Q_k$ of $k$-quotient polymatroids is minor-closed and $k$-dual-closed.
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The class $Q_k$ of $k$-quotient polymatroids is minor-closed and $k$-dual-closed.

Use $r_Q(X \cup e) - r_Q(X) \leq r_L(X \cup e) - r_L(X)$ to get $r_{M_i}$ recursively from $\rho$: $r_{M_i}(\emptyset) = 0$; for $e \in E - X$,

$$r_{M_i}(X \cup e) = \begin{cases} r_{M_i}(X) + 1, & \text{if } \rho(X \cup e) \geq \rho(X) + i, \\ r_{M_i}(X), & \text{otherwise.} \end{cases}$$
The excluded minors for $k$-quotient polymatroids

**Theorem**

Bonin, 2016+

Fix $k \geq 2$. The excluded minors for $Q_k$, within $P_k$, are indexed by the $\binom{k+1}{3}$ 3-subsets $A = \{a, b, c\}$ of $\{0, 1, \ldots, k\}$: if $a < b < c$, define $\rho_A$ on $\{e, f\}$ by $\rho_A(\emptyset) = 0$, $\rho_A(e) = b$, $\rho_A(f) = c$, and $\rho_A(\{e, f\}) = a + c$. 

\[
\begin{align*}
\emptyset & \quad \{e\} & \quad \{f\} & \quad \{e, f\} & \quad a + c - b & \quad a & \quad c > a + c - b > a \\
& \quad b & \quad \{e\} & \quad \{f\} & \quad a & \quad c & \quad b < c
\end{align*}
\]
A sketch of the proof: the easier direction

First: $\rho_A$ is not $r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ for any sequence of quotients. If such a sequence existed, use the recurrence relation for $r_{M_c}$:

- the chain $\emptyset \subseteq \{e\} \subseteq \{e, f\}$ yields $r_{M_c}(\{e, f\}) = 0$;
- the chain $\emptyset \subseteq \{f\} \subseteq \{e, f\}$ yields $r_{M_c}(\{e, f\}) = 1$.

Minimality is obvious by cardinality, so these are excluded minors.
A sketch of the proof: the converse

\[ r_{M_i}(∅) = 0; \text{ if } e \in E - X, \text{ then} \]

\[ r_{M_i}(X \cup e) = \begin{cases} 
  r_{M_i}(X) + 1, & \text{if } \rho(X \cup e) \geq \rho(X) + i, \\
  r_{M_i}(X), & \text{otherwise.} 
\end{cases} \]

Most of the (light) work for the converse is inducting to show that if a \( k \)-polymatroid \( \rho \) has no \( \rho_A \)-minors, then defining \( r_{M_1}, r_{M_2}, \ldots, r_{M_k} \) by the recurrence above is well-defined, i.e., when \( X \cup e = X' \cup e' \), the recurrence gives

\[ r_{M_i}(X \cup e) = r_{M_i}(X' \cup e'). \]

Then check the rank axioms for each \( r_{M_i} \).

It is immediate that \( M_{i+1} \) is a quotient of \( M_i \).
Focus on polymatroids $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ in $D_k$.

An incidence set is a set $X \subseteq E$ with $|X| \geq 2$ and $\rho(X') = |X'| + 1$ for all $X' \subseteq X$ with $1 \leq |X'| \leq 3$.

**Lemma**

All elements of an incidence set $X$ are parallel in one $M_i$. Set $p(X) = M_i$. 

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Structure forced on the matroids in $k$-decompositions
For a graph $G = (V, E)$, with $V = \{v_1, v_2, \ldots, v_n\}$, its Boolean polymatroid is the 2-polymatroid $\rho_G$ on $E$ with $\rho_G(X) = |V(X)|$, where $V(X) = \{v_i : v_i \text{ is incident with at least one edge in } X\}$.

For $i$ with $1 \leq i \leq n$, set $E_i = \{e \in E : e \text{ is incident with } v_i\}$ and $M_i = U_{1,E_i} \oplus U_{0,E-E_i}$.

Thus, for $X \subseteq E$, $\rho_G(X) = r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_n}(X)$. 
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Assume $G$ has no loops. Let $c : V \to \{1, \ldots, \chi(G)\}$ be a coloring.

If $c(v_i) = c(v_j)$, then $E_i \cap E_j = \emptyset$, so we can replace $M_i$ and $M_j$ by $U_{1,E_i} \oplus U_{1,E_j} \oplus U_{0,E-(E_i\cup E_j)}$.
A connection with graph coloring

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Applying this whenever \( c(v_i) = c(v_j) \) gives a decomposition \( \rho_G = r_{N_1} + r_{N_2} + \cdots + r_{N_{\chi(G)}} \) with \( \chi(G) \) terms.
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Note: \( \rho_{C_3} = r_{U_{1,3}} \oplus r_{U_{2,3}} \), and \( 2 < \chi(C_3) \).
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Applying this whenever $c(v_i) = c(v_j)$ gives a decomposition $\rho_G = r_{N_1} + r_{N_2} + \cdots + r_{N_{\chi(G)}}$ with $\chi(G)$ terms.

Note: $\rho_{C_3} = r_{U_{1,3}} \oplus r_{U_{2,3}}$, and $2 < \chi(C_3)$. A fluke!
A connection with graph coloring

Let $[k]$ denote $\{1, 2, \ldots, k\}$.

**Theorem**

Let $G = (V, E)$ be connected, with no loops, and with $|V| \geq 4$.

There is a bijection between colorings $c : V \rightarrow [k]$ and $k$-tuples of matroids $(N'_1, N'_2, \ldots, N'_k)$ on $E$ with $\rho_G = r_{N'_1} + r_{N'_2} + \cdots + r_{N'_k}$.

Thus, $\min\{k : \rho_G \in \mathcal{D}_k\} = \chi(G)$. 

**Corollary**

If $G$ is $(k+1)$-critical with $k \geq 3$, then $\rho_G$ is an excluded minor for $\mathcal{D}_k$, as are its $i$-duals for $2 \leq i \leq k$.

Note: contractions in Boolean polymatroids correspond to a variant on deletions, not contractions, in graphs.

For $k \geq 3$, $(k+1)$-critical graphs are very incompletely understood.
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Bonin, 2016+

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There is a bijection between colorings \(c : V \rightarrow [k]\) and \(k\)-tuples of matroids \((N_1', N_2', \ldots, N_k')\) on \(E\) with \(\rho_G = r_{N_1'} + r_{N_2'} + \cdots + r_{N_k'}\).

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**Corollary**

If \(G\) is \((k + 1)\)-critical with \(k \geq 3\), then \(\rho_G\) is an excluded minor for \(\mathcal{D}_k\), as are its \(i\)-duals for \(2 \leq i \leq k\).

Note: contractions in Boolean polymatroids correspond to a variant on deletions, not contractions, in graphs.

For \(k \geq 3\), \((k + 1)\)-critical graphs are very incompletely understood.
The **chromatic number** of a polymatroid $\rho \in \bigcup_{k \geq 2} D_k$ is
\[
\chi(\rho) = \min\{k : \rho \in D_k\}.
\]

With the predictable definition of direct sums,
\[
\chi(\rho_1 \oplus \rho_2) = \max\{\chi(\rho_1), \chi(\rho_2)\}.
\]

If $\rho'$ is a minor of $\rho$, then $\chi(\rho') \leq \chi(\rho)$.  

*Coloring polymatroids?*
The chromatic polynomial, \( \chi(\rho; k) \), of a polymatroid \( \rho \) is the polynomial that, for \( k \in \mathbb{N} \), gives the number of (ordered) \( k \)-tuples \((M_1, M_2, \ldots, M_k)\) of matroids with \( \rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k} \).

This is a polynomial since it is a sum of multinomial coefficients:

\[
\chi(\rho; k) = \sum_{i=1}^{\rho(E)} \sum_{\substack{M_1, M_2, \ldots, M_i \text{ with } i \text{ matroids all of positive rank} \\text{ such that } a_1 \geq a_2 \geq \cdots \geq a_h}} \left( \begin{array}{c} k \\ a_1, a_2, \ldots, a_h, k - i \end{array} \right)
\]

where the inner sum is over all (unordered) decompositions of \( \rho \) with \( i \) matroids, all of positive rank, and \( a_1 \geq a_2 \geq \cdots \geq a_h \) are the multiplicities of the distinct matroids in the decomposition.

What properties does \( \chi(\rho; k) \) have?
An example

Let $\rho$ be the 3-polymatroid on $E = \{a, b, c, d\}$ in which, for $x, y \in E$, $\rho(x) = 3$, $\rho(\{x, y\}) = 5$, and $\rho(E) = 6 = \rho(E - x)$.

The matroids that make up decompositions of $\rho$:

- $U_{1,4}$, $U_{2,4}$, and $U_{3,4}$ on $E$.
- Four like this: $U_{1,3}$ and $U_{2,3}$ on $E - d$, with $U_{1,2}$ on $\{x, d\}$ for $x \in E - d$. (Complete each with loops.)
- A $U_{1,2}$ on each 2-subset of $E$. (Those on disjoint subsets can be in the same matroid.)

$$
\chi(\rho; k) = k(k - 1)(k - 2)(k - 3)(k - 4)(k - 5) \\
+ 7k(k - 1)(k - 2)(k - 3)(k - 4) \\
+ 3k(k - 1)(k - 2)(k - 3) \\
+ 2k(k - 1)(k - 2) \\
= k^6 - 8k^5 + 18k^4 + 4k^3 - 49k^2 + 34k.
$$

This is not the chromatic polynomial of any graph.
Theorem

Let $G = (V, E)$ be a connected graph with $|V| \geq 4$.

Let $\rho$ be the truncation of $\rho_G$ to rank $t$ where $4 \leq t \leq |V|$.

If $\rho \in \mathcal{D}_k$, then there is a coloring $c : V \to [k]$ of $G$ with
$t \geq \left| \{ i \in [k] : |c^{-1}(i)| = 1 \} \right| + 2 \left| \{ i \in [k] : |c^{-1}(i)| > 1 \} \right|.$

Corollary

For $n \geq 2$, the truncation of $\rho_{C_{2n+1}}$ to rank 4 is an excluded-minor for $\bigcup_{k \geq 2} \mathcal{D}_k$. 
Polymatroids from hypergraphs

A hypergraph is an ordered pair $H = (E, \mathcal{E})$ where $\mathcal{E}$ is a multiset $\{X_1, X_2, \ldots, X_k\}$ of nonempty subsets of $E$.

Each $e \in E$ is a vertex. Each $X_i$ is a (hyper)edge.

For $i \in [k]$, set $M_i = U_{1,X_i} \oplus U_{0,E-X_i}$, a rank-1 matroid with $E - X_i$ as the set of loops.

Let $\rho_H$ be the decomposable polymatroid on $E$ given by

$$\rho_H = r_{M_1} + r_{M_2} + \cdots + r_{M_k}. $$

Thus, for $A \subseteq E$,

$$\rho_H(A) = |\{i \in [k] : A \cap X_i \neq \emptyset\}|.$$
The class of polymatroids from hypergraphs is minor-closed

For $a \in E$, the deletion $(\rho_H)\backslash a$ comes from the hypergraph 
$H\backslash a = (E - a, \mathcal{E}\backslash a)$ where $\mathcal{E}\backslash a = \{X_1 - a, X_2 - a, \ldots, X_k - a\}$
(discarding any copies of the empty set).

The contraction $(\rho_H)/a$ comes from the hypergraph 
$H/a = (E - a, \mathcal{E}/a)$ where $\mathcal{E}/a$ consists of the sets $X_i \in \mathcal{E}$ with $a \notin X_i$.

(These are not ordinary deletion and contraction in hypergraphs; we are deleting or contracting an element of $E$, not $\mathcal{E}$.)
**Proposition**

The polymatroid $\rho_H$ gives the hypergraph $H = (E, \mathcal{E})$.

**Proof:** Use PIE to find, for $X \subseteq E$, the number of $i$ with $X_i = X$.

For $e \in E$ and $i \in [k]$, say $i$ has property $p_e$ when $e \notin X_i$.

Thus, $X_i = X$ iff $i$ has exactly the properties $p_e$ with $e \in E - X$.

For $Y \subseteq E$, the number of integers $i$ lacking at least one property $p_e$, for $e \in Y$, is $\rho_H(Y)$, so $k - \rho_H(Y)$ integers have all of these properties, and maybe more.

Thus, the number of $i$ with $X_i = X$ is

$$\sum_{Y \supseteq E - X} (-1)^{|Y| - |E - X|} (k - \rho_H(Y)) = \sum_{Y \supseteq E - X} (-1)^{|Y| - |E - X| + 1} \rho_H(Y).$$
One can extend these ideas to show that a polymatroid comes from a hypergraph if and only if all sums of the type that arose in the proof above are non-negative.

Proposition

A polymatroid $\rho$ on $E$ is $\rho_H$ for some hypergraph $H$ on $E$ iff

$$\sum_{T \supseteq S} (-1)^{|T-S|+1} \rho(T) \geq 0$$

for all $S \subsetneq E$. 

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The line graph of a hypergraph

For line graph $G_H$ of a hypergraph $H$ has as vertices the edges $X_1, X_2, \ldots, X_k$ of $H$, and an edge joins $X_i$ and $X_j$ whenever $X_i \cap X_j \neq \emptyset$.

For $i \in [k]$, set $M_i = U_{1, X_i} \oplus U_{0, E - X_i}$, a rank-$1$ matroid with $E - X_i$ as the set of loops.

Let $\rho_H$ be the decomposable polymatroid on $E$ given by

$$\rho_H = r_{M_1} + r_{M_2} + \cdots + r_{M_k}.$$

Let $c : E \to [t]$ be a $t$-coloring of $G_H$.

If $c(X_i) = c(X_j)$, then $X_i \cap X_j = \emptyset$. Replace $M_i$ and $M_j$ by $U_{1, X_i} \oplus U_{1, X_j} \oplus U_{0, E - (X_i \cup X_j)}$.

Thus, $\chi(\rho_H) \leq \chi(G_H)$. When is this optimal?
Some sufficient conditions for optimality

**Theorem**

Let $H = (E, \mathcal{E})$ be a hypergraph, where $\mathcal{E} = \{X_1, X_2, \ldots, X_k\}$. Assume that

1. $|X_i \cap X_j| \leq 1$ for all $i, j \in [k]$ with $i \neq j$, and
2. for any three distinct pairwise non-disjoint sets $X_h, X_i, X_j$ in $\mathcal{E}$, some pair of elements in $X_h \cup X_i \cup X_j$ is in no set in $\mathcal{E}$.

If $\rho_H = r_{N_1} + r_{N_2} + \cdots + r_{N_t}$ for matroids $N_1, N_2, \ldots, N_t$ on $E$, then each $N_i$ is a direct sum of uniform matroids of ranks 0 and 1, and the ground sets of the uniform matroids of rank 1 that occur in $N_1, N_2, \ldots, N_t$ are exactly $X_1, X_2, \ldots, X_k$.

For such hypergraphs, the minimal $t$ with $\rho_H \in \mathcal{D}_t$ is $\chi(G_H)$.

This applies to the set $E$ of lines, and subsets $X_i$ of sets of lines through a point, in an affine plane, and many other examples.

Similar results, with different machinery (going further with incidence sets), apply to projective planes and other examples.
Conclusions and a question

We can construct excluded minors for the class $\mathcal{D}_k$ from projective and affine planes, and from many other structures, so finding the excluded minors for $\mathcal{D}_k$ in general is not reasonable.

In particular, giving a complete list of the excluded minors for each $\mathcal{D}_k$ would require settling classical problems like for which orders projective planes exist.

We revealed a direction for extending graph coloring to polymatroids. Can one extend the theory of flows?
We can construct excluded minors for the class $\mathcal{D}_k$ from projective and affine planes, and from many other structures, so finding the excluded minors for $\mathcal{D}_k$ in general is not reasonable. In particular, giving a complete list of the excluded minors for each $\mathcal{D}_k$ would require settling classical problems like for which orders projective planes exist.

We revealed a direction for extending graph coloring to polymatroids. Can one extend the theory of flows?

Thank you for listening.