Minor-Closed Classes of Polymatroids

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Joint work with

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These slides are available at

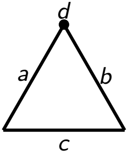
http://blogs.gwu.edu/jbonin/

A polymatroid on a set E is a function $\rho : 2^E \to \mathbb{Z}$ that is

- normalized: $\rho(\emptyset) = 0$,
- non-decreasing: $\rho(A) \leq \rho(B)$ for all $A \subseteq B \subseteq E$, and
- submodular: $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ for all $A, B \subseteq E$.

It is a k-polymatroid if $\rho(e) \leq k$ for all $e \in E$.

Matroids are 1-polymatroids.



A 2-polymatroid: $\rho(\emptyset) = 0$; $\rho(d) = 1$;
sets X with $\rho(X) = 2$: $\{a\}$, $\{b\}$, $\{c\}$, $\{a, d\}$, $\{b, d\}$;
for the rest, $\rho(X) = 3$.

Minors are defined as for matroids, via ρ: for $A \subseteq E$,

- deletion: $\rho_{\setminus A}(X) = \rho(X)$ for $X \subseteq E - A$,
- contraction: $\rho_{/A}(X) = \rho(X \cup A) - \rho(A)$ for $X \subseteq E - A$,
- minors: any combination of deletion and contraction.

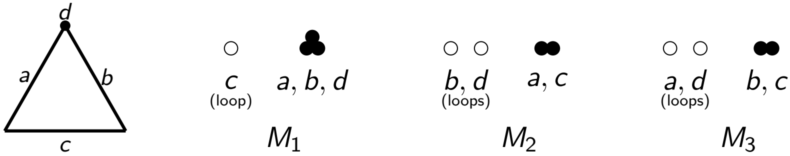
The class $\mathcal{P}_k$ of k-polymatroids is minor-closed.

There is one excluded minor for $\mathcal{P}_k$ for each $t > k$:
ρ on $\{e\}$ with $\rho(e) = t$.

For matroids $M_1, M_2, \ldots, M_k$ on E, defining $\rho : 2^E \to \mathbb{Z}$ by

$$\rho(X) = r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_k}(X),$$

for $X \subseteq E$, gives a k-polymatroid. We say ρ is k-decomposable.



A way to get some k-polymatroids

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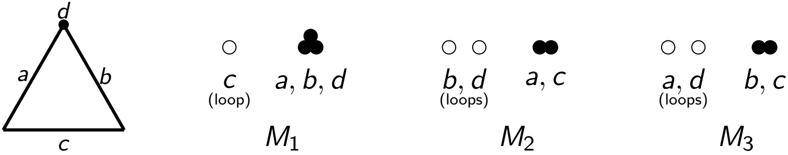

Decompositions interact well with minors:

$$\rho_{\backslash A} = r_{M_1 \backslash A} + r_{M_2 \backslash A} + \cdots + r_{M_k \backslash A}$$
$$\rho_{/A} = r_{M_1 / A} + r_{M_2 / A} + \cdots + r_{M_k / A}.$$

Which polymatroids are of this type?

Which polymatroids are decomposable? (Murty and Simon, 1978.)

Let $\mathcal{D}_k$ be the class of k-decomposable polymatroids.

What are the excluded-minors for $\mathcal{D}_k$? Even $\mathcal{D}_2$ seems to be open.

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Given ρ, there may be many options for $M_1, M_2, \ldots, M_k$.

Lemos (2002) showed how, given one decomposition of a 2-polymatroid ρ, to get all decompositions of ρ.

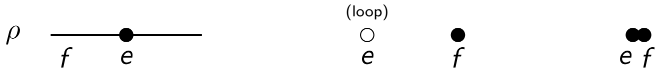
A special case: dc-polymatroids

D. Chun (2009) studied deletion-contraction polymatroids, or dc-polymatroids, that is, 2-polymatroids of the form

$$\rho(X) = r_{M\backslash y}(X) + r_{M/y}(X)$$

for $X \subseteq E$, for some matroid M on $E \cup y$.

A 2-polymatroid that is not a dc-polymatroid:



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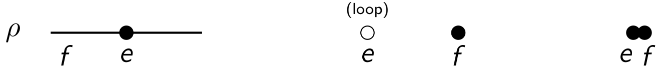

Theorem

The excluded minors for the minor-closed class of dc-polymatroids, within $\mathcal{P}_2$, are ρ (above) and $U_{2,2}$. *(D. Chun, 2009)*

Recall quotients

A matroid Q is a quotient of L, or L is a lift of Q, if $L = M \backslash A$ and $Q = M/A$ for some matroid M and $A \subseteq E(M)$.

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Lemma

For matroids Q and L on E,
Q is a quotient of L if and only if $r_L - r_Q$ is non-decreasing,
that is, for all $X \subseteq Y \subseteq E$,

$$r_L(X) - r_Q(X) \leq r_L(Y) - r_Q(Y),$$

or, equivalently, for all $X \subseteq E$ and $e \in E - X$,

$$r_Q(X \cup e) - r_Q(X) \leq r_L(X \cup e) - r_L(X).$$

A broader special case

A polymatroid ρ of the form $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ where each matroid M_{i+1} is a quotient of M_i is a k-quotient polymatroid.

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The class $\mathcal{Q}_k$ of k-quotient polymatroids is minor-closed.

Use $r_Q(X \cup e) - r_Q(X) \leq r_L(X \cup e) - r_L(X)$.
We get r_{M_i} recursively from ρ: $r_{M_i}(\emptyset) = 0$; for $e \in E - X$,

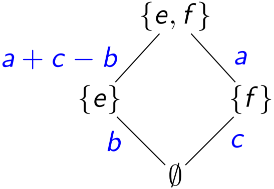
$$r_{M_i}(X \cup e) = \begin{cases} r_{M_i}(X) + 1, & \text{if } \rho(X \cup e) \geq \rho(X) + i, \\ r_{M_i}(X), & \text{otherwise.} \end{cases}$$

Theorem

Fix $k \geq 2$. The excluded minors for $\mathcal{Q}_k$, within $\mathcal{P}_k$, correspond to the $\binom{k+1}{3}$ 3-subsets $A = \{a, b, c\}$ of $\{0, 1, \ldots, k\}$: if $a < b < c$, define ρ_A on $\{e, f\}$ by

$$\rho_A(\emptyset) = 0, \quad \rho_A(e) = b, \quad \rho_A(f) = c, \quad \text{and} \quad \rho_A(\{e, f\}) = a + c.$$

(2017)

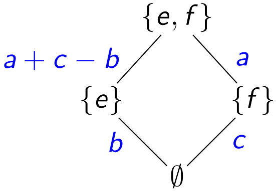


$a < a + c - b < c$

$b < c$

$\rho_{\{0,1,2\}}$

A sketch of the proof: the easier direction



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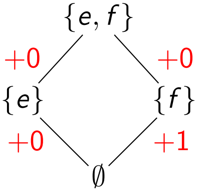


First: ρ_A is not $r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ for any sequence of quotients.
If such a sequence existed, use the recurrence relation for r_{M_c}:
the chain $\emptyset \subseteq \{e\} \subseteq \{e, f\}$ yields $r_{M_c}(\{e, f\}) = 0$;
the chain $\emptyset \subseteq \{f\} \subseteq \{e, f\}$ yields $r_{M_c}(\{e, f\}) = 1$.

Minimality is obvious by cardinality, so these are excluded minors.

A sketch of the proof: the converse

$r_{M_i}(\emptyset) = 0$; if $e \in E - X$, then

$$r_{M_i}(X \cup e) = \begin{cases} r_{M_i}(X) + 1, & \text{if } \rho(X \cup e) \geq \rho(X) + i, \\ r_{M_i}(X), & \text{otherwise.} \end{cases}$$

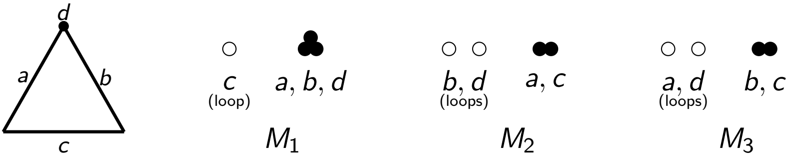
Most of the (light) work for the converse is inducting to show that if a k-polymatroid ρ has no ρ_A-minors, then defining $r_{M_1}, r_{M_2}, \ldots, r_{M_k}$ by the recurrence above is well-defined, i.e., when $X \cup e = X' \cup e'$, the recurrence gives $r_{M_i}(X \cup e) = r_{M_i}(X' \cup e')$.

Then check the rank axioms for each r_{M_i}.

It is immediate that M_{i+1} is a quotient of M_i.

Structure forced on the matroids in k-decompositions

Focus on polymatroids $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ in $\mathcal{D}_k$.



An incidence set is a set $X \subseteq E$ with $|X| \geq 2$ and $\rho(X') = |X'| + 1$ for all $X' \subseteq X$ with $1 \leq |X'| \leq 3$.

Lemma

All elements of an incidence set X are parallel in one M_i.
Set $p(X) = M_i$.

Lemma

Let X and Y be incidence sets with $\rho(\{a, b\}) = 4$ for some $a \in X$ and $b \in Y$. If $X \cap Y \neq \emptyset$, then $|X \cap Y| = 1$ and $p(X) \neq p(Y)$.

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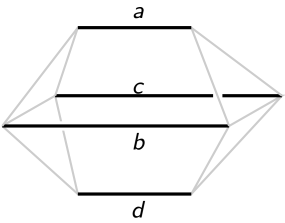
A counterpart of the Vámos matroid: $E = \{a, b, c, d\}$,

$$\rho(X) = \begin{cases} 2\,|X|, & \text{if } |X| \leq 1, \\ 3, & \text{if } |X| = 2 \text{ and } X \neq \{a, d\}, \\ 4, & \text{otherwise.} \end{cases}$$



Set $X = \{a, b, c\}$ and $Y = \{b, c, d\}$ to see that ρ is indecomposable.

It is an excluded-minor for $\mathcal{D}_k$ for all $k \geq 2$.

For a graph $G = (V, E)$, with $V = \{v_1, v_2, \ldots, v_n\}$, its Boolean polymatroid is the 2-polymatroid ρ_G on E with $\rho_G(X) = |V(X)|$, where $V(X) = \{v_i : v_i \text{ is incident with at least one edge in } X\}$.

For i with $1 \leq i \leq n$, set $E_i = \{e \in E : e \text{ is incident with } v_i\}$ and $M_i = U_{1,E_i} \oplus U_{0,E-E_i}$.

Thus, for $X \subseteq E$, $\quad \rho_G(X) = r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_n}(X)$.

Assume G has no loops, and let $c : V \to \{1, 2, \ldots, \chi(G)\}$ be a coloring of G.

If $c(v_i) = c(v_j)$, then $E_i \cap E_j = \emptyset$, so we can replace M_i and M_j by $U_{1,E_i} \oplus U_{1,E_j} \oplus U_{0,E-(E_i \cup E_j)}$.

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Applying this whenever $c(v_i) = c(v_j)$ gives a decomposition $\rho_G = r_{N_1} + r_{N_2} + \cdots + r_{N_{\chi(G)}}$ with $\chi(G)$ terms.

A connection with graph coloring

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Note: $\rho_{C_3} = r_{U_{1,3}} \oplus r_{U_{2,3}}$, and $2 < \chi(C_3)$.

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Note: $\rho_{C_3} = r_{U_{1,3}} \oplus r_{U_{2,3}}$, and $2 < \chi(C_3)$. A fluke!

Let $[k]$ denote $\{1, 2, \ldots, k\}$.

Theorem

Let $G = (V, E)$ be connected, with no loops, and with $|V| \geq 4$.

There is a bijection between colorings $c : V \to [k]$ and k-tuples of matroids $(N'_1, N'_2, \ldots, N'_k)$ on E with $\rho_G = r_{N'_1} + r_{N'_2} + \cdots + r_{N'_k}$.

Thus, $\min\{k : \rho_G \in \mathcal{D}_k\} = \chi(G)$. *(2017)*

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Corollary

If G is $(k+1)$-critical with $k \geq 3$, then ρ_G is an excluded minor for $\mathcal{D}_k$ (as are various duals).

For $k \geq 3$, $(k+1)$-critical graphs are very incompletely understood.

Coloring polymatroids?

The chromatic number of a polymatroid $\rho \in \bigcup_{k \geq 2} \mathcal{D}_k$ is

$\chi(\rho) = \min\{k \,:\, \rho \in \mathcal{D}_k\}$.

With the predictable definition of direct sums,

$$\chi(\rho_1 \oplus \rho_2) = \max\{\chi(\rho_1), \chi(\rho_2)\}.$$

If ρ' is a minor of ρ, then $\chi(\rho') \leq \chi(\rho)$.

Theorem

Let $G = (V, E)$ be a connected graph with $|V| \geq 4$.

Let ρ be the truncation of ρ_G to rank t where $4 \leq t \leq |V|$.

If $\rho \in \mathcal{D}_k$, then there is a coloring $c : V \to [k]$ of G with $t \geq \left|\{i \in [k] : |c^{-1}(i)| = 1\}\right| + 2\left|\{i \in [k] : |c^{-1}(i)| > 1\}\right|$.

(2017)

Corollary

For $n \geq 2$, the truncation of $\rho_{C_{2n+1}}$ to rank 4 is an excluded-minor for $\bigcup_{k \geq 2} \mathcal{D}_k$.

Thank you for listening.