A New Perspective on the $\mathcal{G}$-Invariant of a Matroid

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Joint work with

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These slides and our paper are available at

`http://blogs.gwu.edu/jbonin/`

Let M be a rank-r matroid on $E = \{1, 2, \ldots, n\}$.

The rank sequence of a permutation $\pi = e_1, e_2, \ldots, e_n$ is $\underline{r}(\pi) = r_1, r_2, \ldots, r_n$ where

$$r_i = r(\{e_1, e_2, \ldots, e_i\}) - r(\{e_1, e_2, \ldots, e_{i-1}\}).$$

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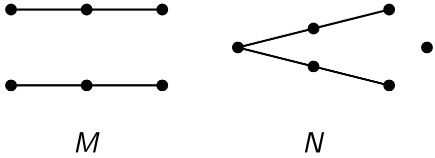
To a sequence $\underline{r}$ of r 1's and $n - r$ 0's, associate a variable $[\underline{r}]$. Let the set of these variables be a basis of a vector space, $\mathcal{G}(n, r)$, over a field of characteristic 0.

The $\mathcal{G}$-invariant is $\mathcal{G}(M) = \sum_{\text{permutations } \pi} [\underline{r}(\pi)]$. (Derksen, 2009; recast)

Example



Two rank sequences:

111000 if $\{e_1, e_2, e_3\}$ is a basis;

there are $\left(\binom{6}{3} - 2\right) \cdot 3! \cdot 3! = 648$ such permutations;

110100 otherwise;

there are $2 \cdot 3! \cdot 3! = 72$ such permutations.

Thus, $\mathcal{G}(M) = \mathcal{G}(N) = 648\,[111000] + 72\,[110100]$.

M and N also have the same Tutte polynomial.

The Tutte polynomial $T(M;x,y)$ of M is

$$\sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} = \sum_{i,j} t_{i,j}(x-1)^{r(E)-j}(y-1)^{i-j},$$

where $t_{i,j}$ is the number of sets A with $|A| = i$ and $r(A) = j$.

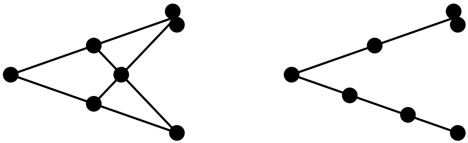
Theorem

We get $T(M;x,y)$ from $\mathcal{G}(M)$:

$$t_{i,j} = \frac{\#\ \text{terms } [\underline{r}] \text{ in } \mathcal{G}(M) \text{ with } r_1 + \cdots + r_i = j}{i!(n-i)!}.$$

(Derksen, 2009)

Two matroids with the same Tutte polynomial but different $\mathcal{G}$-invariants.



The $\mathcal{G}$-invariant is stronger than the Tutte polynomial

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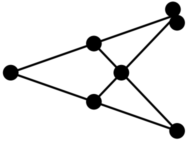
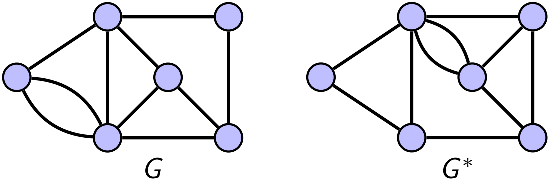


The cycle matroids of the Gray graphs (duals of each other) have the same Tutte polynomial but different $\mathcal{G}$-invariants.



Given a matroid M of rank r on $\{1, 2, \ldots, n\}$, the characteristic vector of each basis is an n-tuple of r ones and $n - r$ zeroes.

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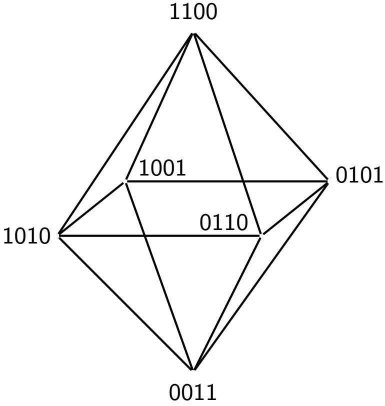
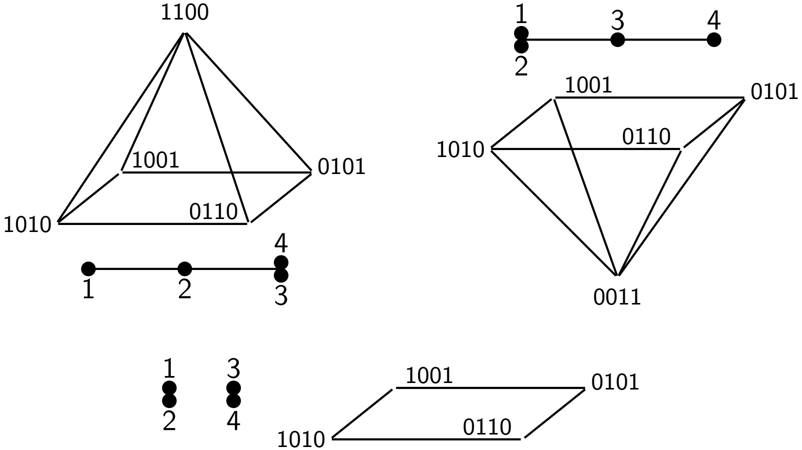
The convex hull of these vectors, over all bases, is the matroid base polytope, $P(M)$, of M.

E.g., for $U_{2,4}$, ,
each 2-element set is a basis, so each 4-tuple with two ones is a vertex of $P(U_{2,4})$.



$P(U_{2,4})$ decomposes into two matroid base polytopes whose intersection is a matroid base polytope of lower dimension.



A subdivision of $P(M)$ is a decomposition of $P(M)$ as a union

$$P(M_1) \cup P(M_2) \cup \cdots \cup P(M_t),$$

where $M_1, M_2, \ldots, M_t$ are matroids on E and,
for each $J \subseteq \{1, 2, \ldots, t\}$ with $|J| \geq 2$, the intersection $\bigcap_{i \in J} P(M_i)$ is $P(M_J)$ for some matroid M_J, and $\bigcap_{i \in J} P(M_i)$ is a proper face of each $P(M_i)$ with $i \in J$.

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A matroid invariant f is a valuation if for any matroid M and the matroids $M_1, M_2, \ldots, M_t$ in any subdivision of $P(M)$, we have

$$f(M) = \sum_{J \subseteq \{1,2,\ldots,t\},\, J \neq \emptyset} (-1)^{|J|+1} f(M_J).$$

Theorem (Universality of the $\mathcal{G}$-Invariant)

Every valuation is a specialization of the $\mathcal{G}$-invariant.

(Derksen and Fink, 2010)

Theorem (The Recipe Theorem)

Fix elements u, v, and units σ, τ, in a commutative ring R with unity. There is a unique R-valued function t on matroids so that

- *if $E = \emptyset$, then $t(M) = 1$,*
- *if e is a loop, then $t(M) = v \cdot t(M \backslash e)$,*

  if e is a coloop, then $t(M) = u \cdot t(M/e)$,

  otherwise, $t(M) = \sigma \cdot t(M \backslash e) + \tau \cdot t(M/e)$,

namely $t(M) = \sigma^{|E|-r(M)} \tau^{r(M)} T(M; \frac{u}{\tau}, \frac{v}{\sigma})$.

(Oxley and Welsh, 1979; earlier versions by Brylawski and, for graphs, Tutte)

A flag of a rank-r matroid M is a maximal chain of flats

$$\mathrm{cl}(\emptyset) = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{r-1} \subset X_r = E(M),$$

and its composition is the sequence $a_0, a_1, \ldots, a_r$ where $a_i = |X_i - X_{i-1}|$.

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If $X_0, X_1, \ldots, X_r$ has composition $a_0, a_1, \ldots, a_r$, then

$$\gamma(a_0, a_1, \ldots, a_r) = \sum_{\pi \text{ giving } X_0, X_1, \ldots, X_r} [\underline{r}(\pi)]$$

depends only on $a_0, a_1, \ldots, a_r$, not M and not $X_0, X_1, \ldots, X_r$.

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The set $\{\gamma(a_0, a_1, \ldots, a_r)\}$, over all compositions, is a basis of $\mathcal{G}(n, r)$, the γ-basis.

Theorem

Let $\nu(M; a_0, a_1, \ldots, a_r)$ *be the number of flags in* M *with composition* $a_0, a_1, \ldots, a_r$. *We have*

$$\mathcal{G}(M) = \sum_{(a_0, a_1, \ldots, a_r)} \nu(M; a_0, a_1, \ldots, a_r)\, \gamma(a_0, a_1, \ldots, a_r).$$

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Two compositions:

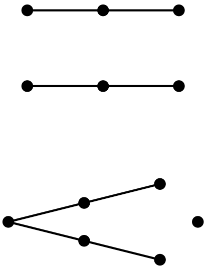
$0, 1, 1, 4$ for chains with a 2-point line;

$\nu(M; 0, 1, 1, 4) = 9 \cdot 2 = 18$;

$0, 1, 2, 3$ for chains with a 3-point line;

$\nu(M; 0, 1, 2, 3) = 2 \cdot 3 = 6$.

$$\mathcal{G}(M) = 18\, \gamma(0, 1, 1, 4) + 6\, \gamma(0, 1, 2, 3)$$



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Thus, each rank-i flat is in $\dfrac{f_r - f_i}{f_{i+1} - f_i}$ flats of rank $i + 1$, so

$$\nu(M; f_0, f_1 - f_0, \ldots, f_r - f_{r-1}) = \prod_{i=0}^{r-1} \frac{f_r - f_i}{f_{i+1} - f_i},$$

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E.g., $\mathcal{G}\big(\mathrm{PG}(r-1, q)\big) = \left(\prod_{i=0}^{r-1} \frac{q^{r-i} - 1}{q - 1}\right) \gamma(0, 1, q, q^2, \ldots, q^{r-1})$.

The effect of some constructions is easy to state using rank-sequences:

- duality: swap 0 and 1, and write in reverse, (Derksen, 2009)
- truncation: change right-most 1 to 0,
- direct sum: use a shuffle product,
- free product (including free extension and coextension): follows from the rank formula for free products.

Others are easy to state using the γ-basis:

- circuit-hyperplane relaxation.

To get a q-cone of a simple $\mathbb{F}_q$-representable rank-r matroid M,

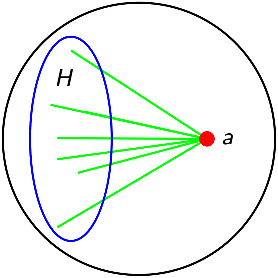
- pick a set E in a hyperplane H of $\mathrm{PG}(r,q)$ with $\mathrm{PG}(r,q)|E$ isomorphic to M,
- pick a point a, the apex, in $\mathrm{PG}(r,q) - H$, and
- restrict $\mathrm{PG}(r,q)$ to the union of the lines $\mathrm{cl}(\{a,e\})$, for $e \in E$.

(Whittle, 1989)

Theorem

For any q-cone M' of M, we can find $\mathcal{G}(M')$ from $\mathcal{G}(M)$, so all q-cones of M have the same $\mathcal{G}$-invariant.

The idea: for a flag $X_0, X_1, \ldots, X_{r+1}$ of M', the $r+1$ sets $\mathrm{cl}_{M'}(X_i \cup a) \cap E$ give a flag of M. We can count how many times each flag of M arises this way.



More data that the $\mathcal{G}$-invariant gives that the Tutte polynomial does not

From the data on flags of all sizes, we get

- the number of saturated chains with given ranks and sizes,
- and so the number of flats of each size and rank,
- and so the number of hyperplanes of each size,
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- and so, dually, the number of circuits of each size.

Also, we can get the number of flats X of rank i, size j, and with k coloops in $M|X$.

Thus, we can get the number of cyclic flats (i.e., flats X so $M|X$ has no coloops) of each size and rank.

Theorem The slicing formula

For k with $0 \leq k \leq r$, let $\mathcal{F}_k$ be the set of rank-k flats in M. Then

$$\mathcal{G}(M) = \sum_{X \in \mathcal{F}_k} \mathcal{G}(M|X) \odot \mathcal{G}(M/X),$$

where $\odot$ is defined by

$$\gamma(a_0, a_1, \ldots, a_{r_1}) \odot \gamma(0, b_1, \ldots, b_{r_2}) = \gamma(a_0, a_1, \ldots, a_{r_1}, b_1, b_2, \ldots, b_{r_2})$$

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on the γ-basis and extended bilinearly.

Corollary

For $0 \leq k \leq r$, we can reconstruct $\mathcal{G}(M)$ from the multiset

$$\{(\mathcal{G}(M|X), \mathcal{G}(M/X)) : X \in \mathcal{F}_k\}.$$

Corollary

If M has girth at least $g+2$, then we can reconstruct $\mathcal{G}(M)$ from the multiset

$$\{\mathcal{G}(M/X) : X \in \mathcal{F}_g\}.$$

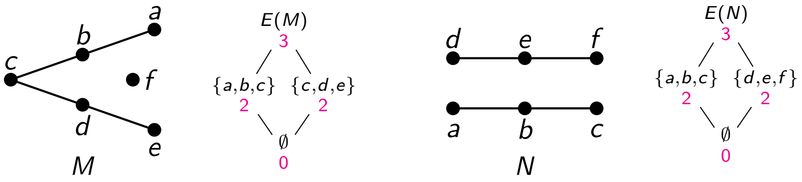
Corollary

We can reconstruct $\mathcal{G}(M)$ from either of the multisets

$$\{\mathcal{G}(M|X) : \text{ hyperplanes } X\} \quad \text{or} \quad \{\mathcal{G}(M/Y) : \text{ circuits } Y\}.$$

Cyclic flats

A set X in a matroid M is cyclic if $M|X$ has no coloops.



The set $\mathcal{Z}(M)$ of cyclic flats of M, ordered by inclusion, is a lattice.

A matroid M is determined by $E(M)$ and the pairs $(X, r(X))$ with $X \in \mathcal{Z}(M)$. (Brylawski, 1975)

Assume M has no coloops.
Its configuration is (L, s, ρ), where L is a lattice, $s : L \to \mathbb{Z}$, $\rho : L \to \mathbb{Z}$, and there is an isomorphism $\phi : L \to \mathcal{Z}(M)$ with $s(x) = |\phi(x)|$ and $\rho(x) = r(\phi(x))$ for all $x \in L$. (Eberhardt, 2014)

Theorem

From the configuration, one can compute the Tutte polynomial.

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The configuration determines the $\mathcal{G}$-invariant

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Matroids with different configurations can have the same $\mathcal{G}$-invariant (e.g., Dowling matroids of the same rank $r > 3$ over non-isomorphic groups of the same order).

Problems

We can compute the $\mathcal{G}$-invariants of:

- perfect matroid designs (e.g., projective & affine geometries),
- $M(K_n)$ and Dowling matroids (recursively),
- paving matroids,
- nested matroid (recursively).

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Thank you for listening.