AN INTRODUCTION TO EXTREMAL MATROID THEORY WITH AN EMPHASIS ON THE GEOMETRIC PERSPECTIVE

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Contents

1. The Scope of These Talks

The goal of these talks is to give you a good idea of what the field of extremal matroid theory is about. We will prove some basic results in extremal matroid theory, we will discuss a number of the many tantalizing open problems that this field offers, and we will present some of the ideas and tools that may prove relevant for the eventual resolution of these questions.

These aims are ambitious and several factors will limit how much we can achieve. One factor is that extremal matroid theory is a very broad subject that encompasses a huge number of results as well as a rich and diverse range of topics. Also, extremal matroid theory can become very technical very quickly. We will therefore take several steps to keep these talks accessible at the introductory level; however, these steps come at the price of limiting the topics we can treat. We will, for

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instance, stress what is most accessible with only modest background in matroid theory; we will also devote some of these talks to providing the necessary background in matroid theory. We will stress one particular perspective, namely, the geometric approach, which is perhaps the easiest perspective to understand on a first encounter with matroid theory. We will stress aspects of the extremal matroid theory whose motivation will be fairly self-evident to a general mathematical audience. At some points we will use the more concrete subject of extremal graph theory as a starting point; we will then pose and study, in the context of matroids, natural counterparts of results in extremal graph theory.

At various points in these talks, we will broaden the scope by sketching additional results in, and other approaches to, extremal matroid theory; to avoid getting too technical, these discussions will indeed be mere sketches. In part, these sketches will serve as pointers for those who wish to pursue the subject in greater depth. Indeed, there is considerable technical machinery that has been developed — much of which comes under the label "matroid structure theory" — that offers powerful tools that can be brought to bear on problems of interest. What we will see here is truly just the very tip of the iceberg.

Thus, we will present selected aspects of extremal matroid theory that have been chosen with the aim of conveying the flavor of the subject at an introductory level; these talks will not provide a comprehensive account of the entire field of extremal matroid theory. I hope that what we lose in breadth we gain in accessibility. I also hope that the material we present will induce you to read further in the literature of this exciting field and to attack some of the many open problems we discuss.

Another way in which these talks will not be an ideal representation of the entire field is that my own work will receive much more attention than is merited either by the number of these results or by their depth. To some extent this is simply a natural pitfall for anyone who is surveying a field in which he or she works. However, I think that there is some justification for this disproportional attention in that several of my papers treat some very fundamental issues in extremal matroid theory and perhaps the very basic nature of these results makes them a more natural and simpler starting point for an introduction to the subject. Also, it is especially in these areas of extremal matroid theory that I can point out more open problems that I believe have not received enough attention; suggesting problems through which you may enter this field is one of the primary aims of these talks.

2. Matroid Theory Background

A thorough introduction to matroid theory alone would require more than ten talks and I want to focus most of these talks on extremal matroid theory. However, since matroid theory is not part of everyone's common mathematical background, I will present some of the essential notions of the subject. For those who have seen matroid theory before, the first several talks should serve as a quick review. At the same time, I will try to give enough background and insight so that those who have never seen matroid theory before will have what they need to follow the later talks. For those who are completely new to matroid theory, I hope that this overview also serves as a guide (and perhaps as motivation) for further reading in the subject.

One of the best starting points for reading more about the basics of matroid theory is the founding paper by Hassler Whitney from 1935, *On the abstract properties of linear dependence* [34]. This remarkable paper includes a surprising amount of the basic material that we will cover in this section. The current standard text by James Oxley [26] is another excellent, and far more comprehensive, starting point that offers a wonderful collection of references.

Throughout these talks, we will focus on *finite* matroids, which account for the vast majority of the matroid literature. About infinite matroids we mention only that there are *several* notions of infinite matroids and some classes of infinite matroids are extremely important; however, some major parts of matroid theory simply do not have satisfactory counterparts in the infinite case. Indeed, trying to resolve these problems is why there are several notions of infinite matroids.

2.1. **Basic Concepts.** Matroid theory is the common generalization of several subjects: linear algebra, graph theory, combinatorial optimization, and more. As is fitting for a subject that generalizes such diverse areas, there are many ways one can approach matroid theory and many concepts that one can take as the starting point. Also, some results that might be difficult to prove from one perspective turn out to be much easier to see from another point of view, so these diverse approaches to matroid theory serve a very practical purpose. So that we will have a useful range of terms to discuss ideas and results in extremal matroid theory, let us start with some of these basic views of a matroid.

The definition (out of the more than 50 equivalent definitions of a matroid) we will start with is motivated by linear algebra and the notion of linear independence.

Definition 2.1. *A* matroid M *is a finite set* S *and a collection* $\mathcal{I}$ *of subsets of* S*, called* independent sets*, such that the following properties hold:*

(i) $\emptyset$ *is in* $\mathcal{I}$*,*
(ii) *if* Y *is in* $\mathcal{I}$ *and* X *is a subset of* Y*, then* X *is in* $\mathcal{I}$*, and*
(iii) *if* X *and* Y *are in* $\mathcal{I}$ *and* $|X| < |Y|$*, then there is an element* x *in* $Y - X$ *such that* $X \cup x$ *is in* $\mathcal{I}$ [The Augmentation Property].

In matroid theory, it is common to use $X \cup x$ as an abbreviation for $X \cup \{x\}$; this and similar short-cuts make expressions easier to read.

Note that if S is a finite set of vectors in a vector space V, then S together with the collection of linearly independent subsets of S forms a matroid. That the first two properties in Definition 2.1 hold is obvious; that the third property holds follows from noting that, by considering the dimensions, the span of X cannot contain all of Y.

A simple but important generalization of the last example is to take as the ground set S of a matroid M the set of indices of the columns of a fixed matrix A over a given field F and to take as the independent sets of M the sets of indices for which the corresponding columns form a linearly independent *set* of vectors over F. We denote this matroid by $M[A]$. There are important differences between this and the previous example. For instance, there is a unique 0-vector in a vector space, that is, a unique vector which by itself is dependent; however, since there can be many zero columns in A, there can be many elements in $M[A]$ each of which, by itself, is dependent. Similarly, for a fixed column A_i of A, if F is finite, then there may be more indices j such that columns A_i and A_j are linear dependent than there might be vectors y such that A_i and y are linearly dependent: since such vectors y are scalar multiples of A_i, the size of the field F puts a bound on the number of such y, but there is no such bound for the columns of a matrix since columns with distinct indices can be identical as vectors.

The matroids that are singled out in the next definition — those that can be viewed as arising from the columns of a matrix as described in the last paragraph — play a fundamental role throughout matroid theory, including in extremal matroid theory.

Definition 2.2. *A matroid M is* representable over a field F *if M is isomorphic to $M[A]$ for some matrix A over F.*

An isomorphism of matroids is exactly what one would expect: a bijection between the ground sets that induces a bijection between the independent sets.

Matroids that are representable over the field of two elements, GF(2), are *binary*; matroids that are representable over GF(3) are *ternary*.

Since matroids that are representable over fields are such natural and fundamental examples of matroids, a central question that has been, and continues to be, the subject of intense research in matroid theory is the following problem: *Which matroids are representable?* This question can be interpreted in several ways. *For which matroids is there a field over which the matroid is representable?* While we known many examples of matroids that cannot be represented over any field, there is no currently known characterization of such matroids. *For a given field F, which matroids are representable over F?* We know the answer to this question for only three fields: GF(2), GF(3), and GF(4). *Which matroids are representable over all fields?* W. T. Tutte solved this problem in the 1950's and we will discuss his characterization of such matroids when we treat Heller's theorem. Although representability is not the focus of these talks, we will say more about this issue as it arises naturally in later talks.

The remarks about columns of matrices that we made before Definition 2.2 suggest some useful notions. A *loop* in a matroid M is an element x for which the set $\{x\}$ is not independent, i.e., $\{x\}$ is dependent. Two nonloops x, y are *parallel* if the set $\{x, y\}$ is dependent. A matroid without loops and without parallel elements is called a *simple matroid*, a *combinatorial geometry*, or simply, a *geometry*.

Recall that the cycles of a graph are the closed paths in which only the first and last vertices are the same. It is an elementary but useful exercise to show that the sets of edges in a graph G that do not contain edge sets of cycles form the independent sets of a matroid on the edge set of G. This is the *cycle matroid* $M(G)$ of the graph G. Thus, $M(G)$ is the matroid in which the ground set is the set $E(G)$ of edges of G and the set of independent sets in $M(G)$ is given as follows:

$$\mathcal{I}\big(M(G)\big) = \{X \,:\, X \subseteq E(G) \text{ and for all circuits } C \text{ of } G, E(C) \not\subseteq X\}.$$

A matroid is *graphic* if it is isomorphic to $M(G)$ for some graph G. The use of the term loop in matroid theory agrees with that in graph theory, namely, an edge that is incident with only one vertex (if e is a loop of the graph, then the set $\{e\}$ is the edge set of a cycle and so is dependent). Likewise, parallel edges in $M(G)$ are parallel edges of the graph, that is, edges incident with the same pair of vertices (such a pair of edges is the edge set of a cycle and so is dependent). Of course it is natural to ask the following question. *Which matroids are graphic?* A nice characterization of these matroids was given by W. T. Tutte; his result is related to Kuratowski's characterization of planar graphs. We will cite Tutte's result later after we have developed some of the notions that the statement of his result uses.

Another fundamental example is the *uniform matroid* $U_{n,m}$ in which the ground set is a fixed m-element set, say $[m] := \{1, 2, \ldots, m\}$, and the independent sets are all subsets of $[m]$ with n or fewer elements. For instance, in $U_{2,4}$ the ground set is $\{1,2,3,4\}$ and the independent sets are the empty set, the four singletons, and the six subsets of two elements. It is immediate to verify that $U_{n,m}$ indeed is a matroid.

We defined matroids in terms of independent sets, so it is natural to talk about *bases*, that is, maximal independent sets. For example, in cycle matroids of connected graphs the bases are the edge sets of spanning trees. In the uniform matroid $U_{n,m}$, the bases are the n-element subsets of $[m]$.

It is an easy consequence of property (iii) in Definition 2.1 that all bases of a given matroid have the same number of elements.

Note that knowing the independent sets of a matroid is equivalent to knowing the bases: the bases are the maximal independent sets and, conversely, the independent sets are the subsets of bases. Thus, it should be possible to characterize matroids in terms of the collection of bases. Indeed, we have the following theorem, which could alternatively be taken as the starting point for matroid theory in place of Definition 2.1.

Theorem 2.3. *A matroid M is a finite set S together with a collection $\mathcal{B}$ of subsets of S such that:*

(i) *$\mathcal{B}$ is nonempty,*
(ii) *no member of $\mathcal{B}$ properly contains another (i.e., $\mathcal{B}$ is an antichain), and*
(iii) *any one of the following three equivalent conditions holds:*
   (a) *for any sets B_1 and B_2 in $\mathcal{B}$ and element x in $B_1 - B_2$, there is an element y in $B_2 - B_1$ such that the set $(B_1 - x) \cup y$ is in $\mathcal{B}$* [Basis Exchange],
   (b) *for any sets B_1 and B_2 in $\mathcal{B}$ and element x in $B_1 - B_2$, there is an element y in $B_2 - B_1$ such that both $(B_1 - x) \cup y$ and $(B_2 - y) \cup x$ are in $\mathcal{B}$* [Symmetric Basis Exchange], *or*
   (c) *for any subsets X and Y of S with $X \subseteq Y$, if there are sets B_1, B_2 in $\mathcal{B}$ with $X \subseteq B_1$ and $B_2 \subseteq Y$, then there is a set B_3 in $\mathcal{B}$ with $X \subseteq B_3 \subseteq Y$* [The Middle Basis Property].

Condition (iii.a) is an obvious consequence of the augmentation property applied to $B_1 - x$ and B_2. Condition (iii.b) is a strengthened (symmetric) form of condition (iii.a). This strengthened form is suggested by syzygies in invariant theory; invariant theory is an interesting source of inspiration for certain topics in matroid theory which was very actively promoted by Gian-Carlo Rota. The idea behind condition (iii.c) is simple: note that X is independent since it is contained in a basis and Y is spanning since it contains a basis, so there should be a basis between X and Y.

Condition (iii.b) merits more attention. It is clear that the sets B_1 and B_2 play symmetric roles in condition (iii.b); however, so do the complements $S - B_1$ and $S - B_2$. Specifically, the statement $x \in B_1 - B_2$ is the same as $x \in (S - B_2) - (S - B_1)$ and the statement $y \in B_2 - B_1$ is the same as $y \in (S - B_1) - (S - B_2)$. Also, we have the two equalities

$$S - \big((B_1 - x) \cup y\big) = \big((S - B_1) - y\big) \cup x \quad \text{and} \quad S - \big((B_2 - y) \cup x\big) = \big((S - B_2) - x\big) \cup y.$$

This may seem trivial, but is it useful. For instance, from these observations we get the following fundamental theorem, which is less transparent from other perspectives.

Theorem 2.4. *Let M be a matroid on the set S and let $\mathcal{B}$ be the collection of bases of M. Then*

$$\mathcal{B}^* := \{S - B \,:\, B \in \mathcal{B}\}$$

is the collection of bases of a matroid on S.

The matroid whose collection of bases is $\mathcal{B}^*$ is called the *dual* of M and it is denoted M^*. Duality is extremely important in matroid theory. The following result is immediate from the definition.

Observation 2.5. *Duality is an involution, that is, for any matroid M we have $(M^*)^* = M$.*

Note that the dual of the uniform matroid $U_{n,m}$ is $U_{m-n,m}$. One can show that the dual of the cycle matroid of a planar graph is the cycle matroid of its dual graph. (For the definition of the dual of a planar graph, see any graph theory text, e.g., [11].) Only planar graphs have dual graphs and a planar graph may have nonisomorphic dual graphs since there may be different planar embeddings of the graph; however, all dual graphs for a given planar graph have the same cycle matroid. Indeed, Whitney's motivation for defining matroid duality was to generalize graph duality, giving *every* graph a dual (as a matroid) even if the graph is not planar. For all graphs, planar or not, the dual of the corresponding graphic matroid can be described in terms of edge-cutsets in the graph; we will say more about this later. The duals of graphic matroids are *cographic matroids*.

Note that the loops of a matroid are precisely the elements that are in no basis of the matroid. The loops of the dual matroid M^* are called the *coloops* or *isthmuses* of M. Thus, x is an isthmus of M if and only if x is in every basis of M.

It is often convenient to extend the notion of bases of a matroid M to bases of arbitrary sets in M; a *basis of a subset* X of the ground set of M is a maximal independent subset of X.

Why is it worth considering bases? Why not work only with independent sets? It turns out that some results are much easier to prove from one perspective than from the other. Indeed, we just saw a simple example; the symmetric basis exchange property naturally leads to matroid duality! One could approach duality from the perspective of independence, but the symmetric setting with bases is simpler and more natural. Indeed, having discovered duality by considering bases, we could now deduce a theorem about independence from this; however, we will not pursue this.

There are over a half-dozen basic concepts like independent sets and bases that we need to consider and accompanying these notions are about a dozen equivalent ways of defining a matroid that are important to have in mind. Dealing with these different perspectives is an essential skill for those working in matroid theory; it is very important to look at every problem in a half dozen different ways. To keep these talks accessible though, I will try to minimize this. Still, some ability to work with these different perspectives will be important. Thus, we will consider more of these basic concepts and equivalent ways of defining a matroid. I will motivate the connections between these notions. Since the goal of these talks is to introduce you to extremal matroid theory rather than just matroid theory, I will omit the proofs of the equivalence of these different approaches to matroid theory (as we have omitted the proof of Theorem 2.3). For the proofs, see [26] or, better yet, prove the equivalence as an exercise. (Most of the proofs are fairly easy.)

As suggested by the terminology we have already used, and consistent with linear algebra, sets that are not independent are *dependent*. We could develop an approach to matroid theory based on dependent sets, but this will not be important for what we will do. Far more important are the *minimal* dependent sets. Recall that in graphic matroids the independent sets are the sets of edges that do not contain the edge sets of cycles. Cycles are sometimes called circuits and this is the term we adopt in matroids in general.

Definition 2.6. *The* circuits *of a matroid M are the minimal dependent sets.*

Thus, the circuits in graphic matroids are the edges sets of cycles in the corresponding graphs. Also, the circuits of the uniform matroid $U_{n,m}$ are the sets of size $n+1$, which exist if $n < m$. The uniform $U_{n,n}$ has no circuits, and thus no dependencies, and so is called the *free matroid on n elements*.

Note that x is a loop of M if and only if $\{x\}$ is a circuit of M. Also, elements x and y of M are parallel if and only if $\{x, y\}$ is a circuit of M.

Note that the circuits determine the dependent sets (the supersets of circuits), and hence the independent sets (sets that are not dependent), and hence the matroid. Thus, we should be able to characterize matroids in terms of their collections of circuits, that is, we should be able to characterize which collections of sets are the collections of circuits of a matroid. The next theorem gives two such characterizations.

Theorem 2.7. *A matroid M is a finite set S together with a collection $\mathcal{C}$ of subsets of S such that:*

(i) *$\emptyset$ is not in $\mathcal{C}$,*
(ii) *no member of $\mathcal{C}$ properly contains another (i.e., $\mathcal{C}$ is an antichain), and*
(iii) *either of the following equivalent conditions holds:*
 (a) *for any distinct sets C_1 and C_2 in $\mathcal{C}$ and any element x in $C_1 \cap C_2$, there is a set C in $\mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) - x$* [Circuit Elimination]*, or*
 (b) *for any distinct sets C_1 and C_2 in $\mathcal{C}$, any element x in $C_1 \cap C_2$, and any element y in $C_1 - C_2$, there is a set C in $\mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) - x$ and y is in C* [Strong Circuit Elimination].

The circuit elimination properties are motivated by graph theory, in which setting the properties are transparent (see Figure 1). The circuit elimination properties are also easy to see directly for matroids that are representable over a field.

Another basic concept from linear algebra is that of dimension. The dimension of a set of vectors is the largest cardinality among linearly independent subsets of the set. (In linear algebra, dimension is usually defined only for subspaces but there is no need to consider only subspaces.) When stated this way it is clear that dimension is a matroid concept. Since the matroid counterpart of dimension has slightly weaker properties than does the notion of dimension in linear algebra, we give it a different name: rank. Thus, the *rank* $r(X)$ of a set X in a matroid M is the maximal cardinality among independent subsets of X, that is,

$$r(X) = \max\{|I| \,:\, I \in \mathcal{I} \text{ and } I \subseteq X\}.$$

Equivalently, the rank of a set is the cardinality of a basis of the set.

Note that in the uniform matroid $U_{n,m}$, the rank function is given by

$$r(X) = \min\{|X|, n\}.$$

In the graphic matroid $M(G)$, the rank function is given by $r(X) = m - \omega(X)$ where G has m vertices and $\omega(X)$ is the number of components in the graph $G[X]$ with edge set X and the same vertex set as G; to justify this expression for $r(X)$ note that a basis of X consists of a spanning tree for each component of $G[X]$, and a tree on n vertices has $n-1$ edges.

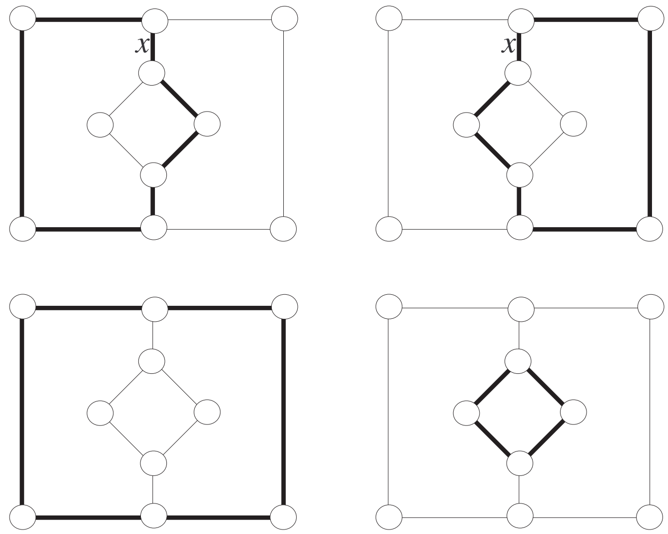

FIGURE 1. Two graph circuits C and C' that both contain x, along with two of the circuits in $(C \cup C') - x$.

The rank $r(M)$ of a matroid M is the rank $r(S)$ of its ground set. Thus, the rank of a matroid is the size of any basis of the matroid. Recall that the complement, relative to the ground set, of any basis of the matroid M is a basis of the dual matroid. In this way we get the following useful result.

Observation 2.8. *For a matroid M and its dual M^* on the ground set S, we have the equality $r(M) + r(M^*) = |S|$.*

A short argument using the definitions of rank and duality gives the following theorem. This result often provides the simplest approach to proving many properties about duality.

Theorem 2.9. *Let M be a matroid on the ground set S with rank function r. The rank function r^* of the dual matroid M^* is given by*

$$r^*(A) = |A| - r(M) + r(S - A).$$

Note that the rank function of a matroid completely captures the matroid since it captures the independent sets: a set X is independent if and only if it is its own largest independent subset, that is, if and only if $r(X) = |X|$. Thus, one can characterize matroids by characterizing which functions are the rank functions of matroids.

Theorem 2.10. *A matroid M is a finite set S together with a function $r : 2^S \to \mathbb{Z}$ such that:*

(i) $0 \leq r(X) \leq |X|$ *for all subsets X of S* [Cardinality Bound],
(ii) *if $X \subseteq Y \subseteq S$, then $r(X) \leq r(Y)$* [Monotonicity], *and*
(iii) $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ *for all subsets X and Y of S* [Semimodularity].

As with the properties of bases or circuits, the first two properties of rank functions are obvious from the definition. The third property requires just a bit of work to see. Since our focus is not the axiom systems, we are omitting the proofs of such results. However, we should note that semimodularity is a weakening of the familiar dimension theorem of linear algebra:

$$\dim(X) + \dim(Y) = \dim(X \cap Y) + \dim(X + Y) \tag{1}$$

for subspaces X and Y. (Of course, $X \cup Y$ spans the subspace sum $X + Y$ and the subspace sum $X + Y$ typically does not make sense in an arbitrary matroid.)

Above we observed that a set X in a matroid M is independent if and only if $r(X) = |X|$. Thus, a set C is a circuit of M if and only if $r(C) < |C|$ and for every x in C we have $r(C - x) = |C| - 1$.

Rank properties are a bit more delicate that the properties for bases or circuits in that there are several basic rank properties but they cannot be mixed at will to obtain characterizations the rank functions of matroids. (Thus, we cannot state a theorem quite like Theorem 2.3 with three options for the last property.) Here is another formulation of matroids in terms of the rank function.

Theorem 2.11. *A matroid M is a finite set S together with a function $r : 2^S \to \mathbb{Z}$ such that:*

(i) $r(\emptyset) = 0$ [Normalization],
(ii) *if $X \subseteq S$ and $x \in S$, then $r(X) \leq r(X \cup x) \leq r(X) + 1$* [Unit Increase], *and*
(iii) *for $X \subseteq S$ and $x, y \in S$, if $r(X \cup x) = r(X)$ and $r(X \cup y) = r(X)$, then $r(X \cup \{x, y\}) = r(X)$* [Local Semimodularity].

Subspaces of a vector space are usually defined via algebraic properties: subspaces are the sets of vectors that are closed under vector addition and scalar multiplication. However, subspaces can also be defined by using the dimension function: a subspace is a set A of vectors that is maximal with respect to having a given dimension, that is, $\dim(A \cup x) > \dim(A)$ for all vectors x that are not in A. This formulation suggests a matroid counterpart. Thus, we define a *flat* of a matroid M to be a subset A of the ground set of M such that $r(A \cup x) > r(A)$ for all elements x of the matroid that are not in A.

Shortly we will see that the rank function of a matroid can be determined from the collection of flats. Thus, there should be characterizations of matroids in terms of their collections of flats. This is the content of the next theorem.

Theorem 2.12. *A matroid M is a finite set S and a collection $\mathcal{F}$ of subsets of S such that the following properties hold:*

(i) *S is in $\mathcal{F}$,*
(ii) *if X and Y are in $\mathcal{F}$, then $X \cap Y$ is in $\mathcal{F}$, and*
(iii) *if X is a proper subset of S that is in $\mathcal{F}$ and if x is in $S - X$, then there is precisely one flat Y with the following property: $X \cup x \subseteq Y$ and there is no set Z in $\mathcal{F}$ with $X \subsetneq Z \subsetneq Y$.*

There are several things to note about this theorem. By property (ii) the collection of flats of a matroid M is closed under intersection; by property (i) there is a largest flat, namely the ground set S; it follows from these statements and basic results in the theory of ordered sets that the collection of flats of a matroid forms a lattice under inclusion. This lattice is the *lattice of flats of M*. Occasionally the lattice theory perspective will be useful for what we do. Here we simply note some

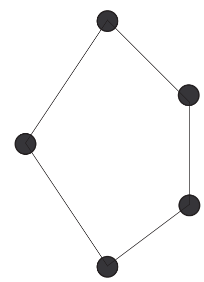

FIGURE 2. The Hasse diagram of a lattice that is not ranked.

terminology from lattice theory that we will use: we say that a flat X is *covered by* a flat Y, or that the flat Y *covers* or *is a cover of* X, if $X \subsetneq Y$ and there no flat Z with $X \subsetneq Z \subsetneq Y$.

We also note that the lattice of flats of a matroid is a ranked lattice, that is, a lattice for which there is a well-defined function (the rank function) that is zero on the least element and such that the rank of any cover of an element is one greater than the rank of that element. (Ranked lattices stand in contrast to lattices such as the one whose Hasse diagram appears in Figure 2.) Note that the covering relation in the lattice of flats of a matroid can be captured in terms of bases of flats as follows: for flats X and Y of a matroid M, the flat Y covers X if and only if for any basis B of X and any element y of $Y - X$, the set $B \cup y$ is a basis of Y. It follows that the rank function of the lattice of flats of a matroid is exactly the restriction of the matroid rank function to the flats. In turn it follows, as we suggested above, that one can recover the rank function of a matroid, and hence the independent sets, from the flats since by local semimodularity and the notion of a flat each set A is contained in a smallest flat, namely

$$\mathrm{cl}(A) = \{x \,:\, r(A) = r(A \cup x)\},$$

and the sets A and $\mathrm{cl}(A)$ have the same rank. The set $\mathrm{cl}(A)$ is call the *closure* of A.

It is easy to see that in the uniform matroid $U_{n,m}$ on the set S we have

$$\mathrm{cl}(X) = \begin{cases} X, & \text{if } |X| < n; \\ S, & \text{otherwise.} \end{cases}$$

From the description we gave above of the rank function of a graphic matroid $M(G)$, it follows that in $M(G)$ the closure $\mathrm{cl}(X)$ of a set X of edges consists of all edges that lie entirely within the components of the induced subgraph on X; that is, an edge e is in $\mathrm{cl}(X)$ if and only if the vertices incident with e are in the same component of the induced graph on the edge set X. This assertion follows since the ranks $r(X)$ and $r(X \cup e)$ are equal if and only if the induced subgraphs on X and $X \cup e$ have the same number of components. It follows that the flats of the complete graph K_n can be identified with the partitions of $[n]$ and the lattice of flats of the cycle matroid $M(K_n)$ is isomorphic to the partition lattice Π_n.

Note that the flats of a matroid M are precisely the images of its closure operator cl. Thus, the closure operator cl determines the flats and so determines the matroid.

Therefore we should be able to characterize matroids in terms of their closure operators. This is done in the next theorem.

Theorem 2.13. *A matroid M is a finite set S and a function* $\mathrm{cl} : 2^S \to 2^S$ *such that the following properties hold:*

(i) $X \subseteq \mathrm{cl}(X)$ *for every subset X of S,*
(ii) *if $X \subseteq Y \subseteq S$, then* $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$*,*
(iii) $\mathrm{cl}\big(\mathrm{cl}(X)\big) = \mathrm{cl}(X)$ *for every subset X of S, and*
(iv) *for any subset X of S and any pair of elements x and y in S, if y is in* $\mathrm{cl}(X \cup x) - \mathrm{cl}(X)$*, then x is in* $\mathrm{cl}(X \cup y) - \mathrm{cl}(X)$ [The Steinitz-MacLane Exchange Property].

To some extent a matroid can be viewed as a special type of lattice, specifically a geometric lattice (i.e., an atomic, semimodular lattice). However, the lattice of flats considered merely as a lattice (without keeping track of the flats as sets) does not capture all information about a matroid; specifically, all information about loops and parallel elements is lost and from many perspectives (duality, for instance) this information is essential. Thus, matroid duality, for instance, is *not* a lattice operation. One can show that there is a bijection between the *simple* matroids on n elements and the geometric lattices with n atoms. However, even with this bijection, the lattice theory perspective plays a useful but very limited role in matroid theory.

An essential way of getting intuition for matroid theory is to think about geometric representations of matroids of small rank in terms of points, lines, and planes. The points, lines, and planes are simply the flats of ranks 1, 2, and 3. Specifically, we draw the rank-1 flats of the matroid (which we call the *points*, as opposed to the elements of the matroid) as geometric points, with dots on top of dots, if needed, to indicate parallel elements; we draw rank-2 flats as lines (not necessarily straight) and we call them *lines*; similarly, rank-3 flats are drawn as, and called, *planes*. Loops are often drawn as hollow dots off to the side. There are several conventions in these drawings that simplify the pictures: if two points are the only points on a line, the line is not drawn (it is understood to be a line; such lines are called *trivial lines*); similarly, if a plane in a matroid of rank greater than three contains only three points, it is not drawn explicitly as a plane. Simple examples are shown in Figures 3 and 4. It is a good exercise to identify the independent sets, the bases, and the rank function in concrete examples such as these. The example in Figure 3 has three trivial lines, and the whole matroid is a plane. Obviously such drawings are limited to low ranks but the intuition one can get from them is invaluable.

There are several commonly-used terms for certain flats in addition to points, lines, and planes. Consistent with the terminology of linear algebra, flats of rank $n-1$ in a matroid of rank n are called *hyperplanes*. The lattice theory perspective (with its terminology of atoms and coatoms) gave rise to the alternative term *copoints* for hyperplanes. Sometimes it is convenient to extend this terminology to flats of rank $n-2$ and $n-3$ in a matroid of rank n by calling such flats *colines* and *coplanes*, respectively. Sometimes a flat of rank $n-i$ in a matroid of rank n is said to have *corank* i. (The last three terms are not universally used but we will sometimes find them convenient.)

Many sources, e.g., [26], reserve the prefix "co" for objects related by duality. For instance, the bases of the dual matroid M^* of M are called the *cobases* of M and the circuits of M^* are called the *cocircuits* of M.

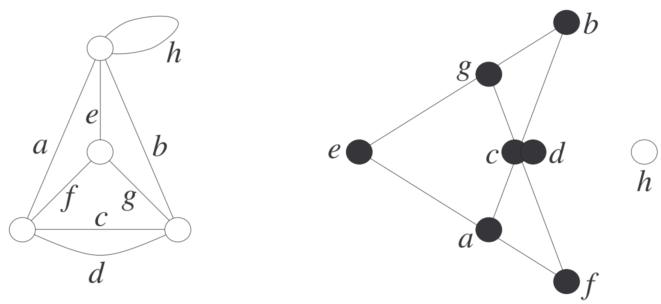

FIGURE 3. The graph K_4 with one pair of parallel edges and one loop added; the geometric representation of the corresponding cycle matroid.

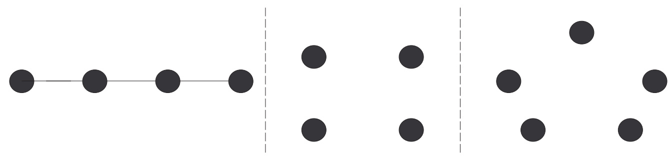

FIGURE 4. The uniform matroids $U_{2,4}$, $U_{3,4}$, and $U_{3,5}$.

Cocircuits are important in part because in addition to the significance they have as the circuits of the dual matroid, they also have an important direct geometric meaning in the matroid itself. The circuits of M are the minimal sets that are contained in no basis of M. Thus, the cocircuits of M are the minimal sets that are contained in no complement of a basis of M. Thus, the complements of the cocircuits of M are the maximal sets that contain no basis of M. Of course, such sets are simply the hyperplanes of M. Thus, we get the following useful result.

Observation 2.14. *The cocircuits of M are the complements of the hyperplanes of M.*

For example, the hyperplanes of the uniform matroid $U_{n,m}$, where $n > 0$, are the subsets of $[m]$ of size $n-1$; thus the cocircuits of $U_{n,m}$ are the subsets of $[m]$ of size $m-n+1$. We noted earlier that each flat of a graphic matroid $M(G)$ induces a partition of the vertex set of the graph G; the blocks of the partition are the components of the induced subgraph. It follows that the hyperplanes of a graphic matroid $M(G)$ correspond to such partitions in which the corresponding induced subgraph of G has one more component than G. Thus, the cocircuits of a graphic matroid $M(G)$ are the minimal edge sets whose removal from G increases the number of components. Thus, the cocircuits of a graphic matroid $M(G)$ are what many graph theorists would call the *minimal edge-cutsets* of G. The circuits of a cographic matroid $M^*(G)$ are therefore also the minimal edge-cutsets of G.

Note that the loops of a matroid are in all flats of the matroid; in particular, the loops are in all hyperplanes of the matroid. Conversely, if x is not a loop, then the set $\{x\}$ is independent and so can be extended to a basis B of the matroid and the element x is not in the hyperplane $\mathrm{cl}(B-x)$. Thus, an element x of a matroid M

is a loop of M if and only if x is in all hyperplanes of M. Equivalently, an element x of M is a loop if and only if x is in no cocircuit of M. By duality, we get the following useful observation.

Observation 2.15. *An element x of a matroid M is an isthmus of M if and only if x is in no circuit of M.*

2.2. **New Matroids from Old.** There are many ways to obtain other matroids from a given matroid or set of matroids; we will examine a few of the most basic constructions of this type, all of which will play important roles later in these talks. The first construction is suggested by the subgraph induced on a given set of edges.

Definition 2.16. *Let M be a matroid on the set S and let T be a subset of S. The* restriction *$M|T$ of M to T is the matroid on T that has as independent sets the subsets of T that are independent in M.*

For those who are new to matroid theory, it would be a useful (and very easy) exercise to prove that the independent sets of the restriction $M|T$ indeed satisfy the conditions in Definition 2.1. Proving the following basic theorem is another useful exercise.

Theorem 2.17. *Let T be a subset of the ground set of a matroid M.*

(i) *The rank function of $M|T$ is the rank function of M restricted to the subsets of T.*
(ii) *The circuits of $M|T$ are the circuits of M that are subsets of T.*
(iii) *The flats of $M|T$ are the sets of the form $F \cap T$ as F ranges over the flats of M.*
(iv) *The closure operator $\mathrm{cl}_{M|T}$ of $M|T$ is given by $\mathrm{cl}_{M|T}(X) = \mathrm{cl}(X) \cap T$.*

The notation $\mathrm{cl}_{M|T}$ in part (iv) of this theorem illustrates how we will use subscripts when they help to clarify to which matroid a closure operator, rank function, etc., refers.

If the focus is instead on what is being removed, the restriction $M|T$ is called the *deletion* $M\backslash(S-T)$.

Recall that an element x is an isthmus of a matroid M if and only if x is in every basis of M. From the definition of the rank of a matroid, it follows that an element x is an isthmus of a matroid M if and only if $r(M\backslash x) = r(M) - 1$. From Observation 2.15 and part (ii) of Theorem 2.17 it follows that if x is an isthmus of M and x is in the restriction $M|T$, then x is also an isthmus of $M|T$.

The matroid in Figure 5 shows the restriction of that in Figure 3 to the set $\{a, b, c, e, f, g\}$; alternatively, the matroid in Figure 5 is obtained by deleting d and h from that in Figure 3. The matroid in Figure 5 is a simple matroid that is obtained by deleting all loops and all but one element in each point of the matroid in Figure 3. This is an important matroid operation; the result is the simplification $\mathrm{si}(M)$ of the matroid M.

Definition 2.18. *Let M be a matroid on a set S and let T be a subset of S that contains no loops and precisely one element of each rank-1 flat. The restriction $M|T$ is the* simplification *$\mathrm{si}(M)$ of M.*

It is easy to see that up to isomorphism a matroid has a unique simplification, so calling $\mathrm{si}(M)$ *the* simplification of M is appropriate. It is also easy to see that a matroid and its simplification have isomorphic lattices of flats.

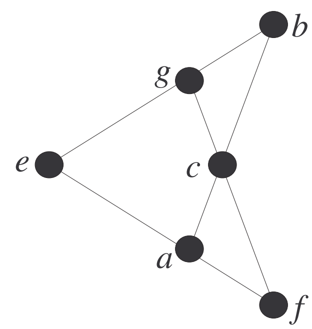

FIGURE 5. The cycle matroid $M(K_4)$ as a restriction of the matroid in Figure 3 to $\{a, b, c, e, f, g\}$.

From part (iii) of Theorem 2.17 it follows that if F is a flat of a matroid M, then the restriction $M|F$ essentially corresponds to the lower interval $[\mathrm{cl}(\emptyset), F]$ in the lattice of flats, that is, the collection of flats contained in F ordered in the same way, i.e., by inclusion. This suggests the following question. What matroid operation corresponds to the upper interval $[F, S]$? This is the operation of contraction, which we turn to next. Just as restriction is more general than lower intervals in the lattice of flats (since we do not have to restrict to a flat), so contraction is more general than upper intervals, but the intuition gained by looking at upper intervals is a reliable guide for the operation of contraction in general.

The flats in an upper interval $[F, S]$ of the lattice of flats of a matroid M are the flats of M that contain the flat F. These flats are ordered by inclusion. Since F is the least of these flats, all elements in F would necessarily be loops in any matroid for which the interval $[F, S]$ is to be the lattice of flats so it is reasonable to eliminate the elements of F. Thus, the flats of the matroid we would like to have correspond to the upper interval $[F, S]$ should be the sets of the form $Z - F$ as Z ranges over the flats of M that contain F. These ideas motivate the following definition.

Definition 2.19. *Let M be a matroid on the set S and let X be a subset of S. The* contraction *M/X of M by X is the matroid on $S - X$ that has as flats the sets of the form $F - X$ as F ranges over the flats of M that contain X.*

In the lattice of subspaces of a vector space, upper intervals essentially correspond to quotients of the vector space. From this perspective we see that, complementary to the operation of restriction which gives a type of subobject, the operation of contraction gives a type of quotient. (However, in matroid theory we reserve the term quotient for a different construction, one that will not enter into these talks.)

It is not difficult to check that the flats described in Definition 2.19 indeed satisfy the properties of flats for a matroid in Theorem 2.12.

Note that if x is an element of M that is not a loop, then the points of the contraction M/x are the lines of M that contain x, with x removed. In general, the rank-i flats of the contraction M/x are the rank-$(i+1)$ flats of M that contain x, with x removed. In this sense, the contraction M/x is a geometric projection through the point x. This is illustrated in Figure 6 in which we contract the matroid in Figure 3 by the point d. In this figure, the flats of M that contain d are the following:

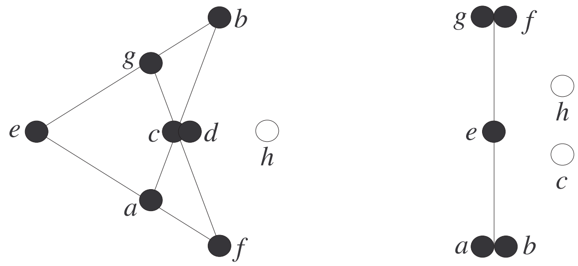

FIGURE 6. The matroid M and the contraction M/d.

$\{c, d, h\}$, $\{c, d, e, h\}$, $\{a, b, c, d, h\}$, $\{c, d, f, g, h\}$, and $\{a, b, c, d, e, f, g, h\}$. Therefore the flats of the contraction M/d are $\{c, h\}$, $\{c, e, h\}$, $\{a, b, c, h\}$, $\{c, f, g, h\}$, and $\{a, b, c, e, f, g, h\}$.

The following facts about contractions are not difficult to prove; doing so is a useful exercise.

Theorem 2.20. *Let X be a subset of the ground set S of a matroid M.*

(i) *The independent sets of the contraction M/X are the subsets I of $S - X$ such that for some (equivalently, every) basis B of the restriction $M|X$, the set $I \cup B$ is independent in M.*
(ii) *The bases of the contraction M/X are the subsets B' of $S - X$ such that for some (equivalently, every) basis B of the restriction $M|X$, the set $B' \cup B$ is a basis of M.*
(iii) *The circuits of M/X are the minimal nonempty sets of the form $C - X$ as C ranges over the circuits of M. Also, if X is an independent set of M and C is a circuit of M that contains X, then $C - X$ is a circuit of M/X.*
(iv) *The rank function of M/X is given by*

$$r_{M/X}(A) = r(A \cup X) - r(X).$$

(v) *The closure operator of M/X is given by*

$$\mathrm{cl}_{M/X}(A) = \mathrm{cl}(A \cup X) - X.$$

An important connection between the operations of deletion and contraction that at first may be surprising is given the following theorem.

Theorem 2.21. *Deletion and contraction are dual operations in the following sense:*

$$M/X = (M^*\backslash X)^* \qquad \textit{and} \qquad M\backslash X = (M^*/X)^*.$$

The equalities $M\backslash X\backslash X' = M\backslash(X \cup X')$ and $M/X/X' = M/(X \cup X')$ clearly hold for any disjoint subsets X and X' of the ground set of M. One can also check that deletion and contraction commute in the following sense. (This theorem is particularly easy to prove by using the rank functions.)

Theorem 2.22. *If X and Y are disjoint subsets of the ground set of M, then $M\backslash X/Y = M/Y\backslash X$.*

One of the fundamental notions in many of these talks will be that of minors of matroids. The minors of a matroid M are formed by any combination of deletions

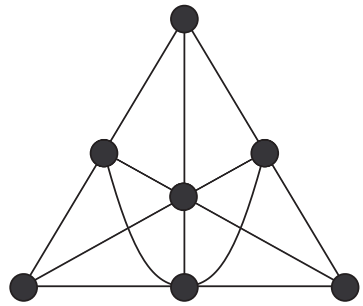

FIGURE 7. The Fano plane, F_7.

and contractions in M. By Theorem 2.22 and the comments before that theorem, minors can always be written as a single deletion followed by a single contraction.

Definition 2.23. *A* minor *of a matroid M is any matroid of the form $M\backslash X/Y$ where X and Y are disjoint (possibly empty) subsets of the ground set of M.*

The term contraction comes from the interpretation of this operation for graphic matroids. For graphic matroids, contracting a set of elements in $M(G)$ yields the cycle matroid of the graph obtained by contracting the corresponding edges in G in the graph-theoretic sense, that is, for each edge contracted we identify the endpoints and delete the edge. Thus, the class of graphic matroids is a *minor-closed class*, that is, minors of matroids in this class are also in this class.

A large part of the interest in matroid minors is due to the following ideas. A minor-closed class of matroids can be characterized by its *excluded minors*, that is, the minor-minimal matroids that are not in the class. Thus, the excluded minors are the minimal obstructions to being in the class. We illustrate this idea by citing (without proof) the following theorem of W. T. Tutte from 1959 [33]. The matroid F_7, which appears in this theorem and will play an important role in later talks, is shown in Figure 7. This matroid is also known as the *Fano plane* or the projective plane $\mathrm{PG}(2,2)$ of order 2.

Theorem 2.24. *A matroid is graphic if and only if it has no minor isomorphic to any of the following matroids: $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, and $M^*(K_{3,3})$.*

Thus, there are five excluded minors for the class of graphic matroids; the last two are related to Kuratowski's characterization of planar graphs.

We have discussed three ways of getting matroids from given matroids, namely, via duality, deletion, and contraction. There are a number of other such constructions, perhaps the most basic of which is direct sums.

Definition 2.25. *Let M_1 and M_2 be two matroids on the disjoint ground sets S_1 and S_2 with collections of independent sets $\mathcal{I}_1$ and $\mathcal{I}_2$. The* direct sum *$M_1 \oplus M_2$ of M_1 and M_2 is the matroid on the union $S_1 \cup S_2$ in which the collection of independent sets is given by*

$$\mathcal{I}(M_1 \oplus M_2) = \{I_1 \cup I_2 \,:\, I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}.$$

It is an easy exercise to prove the following theorem.

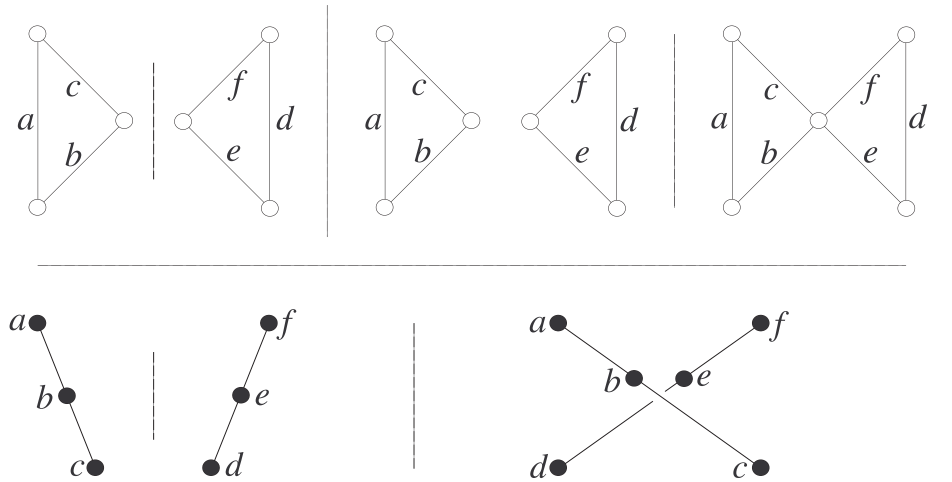

FIGURE 8. Two copies of K_3, with the uniform matroid $U_{2,3}$ as the cycle matroid of each; two graphs whose cycle matroid is the direct sum $U_{2,3} \oplus U_{2,3}$; the corresponding geometric representations of the two copies of $U_{2,3}$ and their direct sum.

Theorem 2.26. *Let M_1 and M_2 be two matroids on the disjoint ground sets S_1 and S_2.*

(i) *The bases of $M_1 \oplus M_2$ are the unions of the form $B_1 \cup B_2$ where B_1 is a basis of M_1 and B_2 is a basis of M_2.*
(ii) *The set of circuits of $M_1 \oplus M_2$ is the union of the set of circuits of M_1 and the set of circuits of M_2.*
(iii) *The rank function of $M_1 \oplus M_2$ is given by*

$$r_{M_1 \oplus M_2}(X) = r_{M_1}(X \cap S_1) + r_{M_2}(X \cap S_2).$$

(iv) *The flats of $M_1 \oplus M_2$ are the unions of the form $F_1 \cup F_2$ where F_1 is a flat of M_1 and F_2 is a flat of M_2.*
(v) *The lattice of flats of $M_1 \oplus M_2$ is the direct product (in the sense of ordered sets) of the lattices of flats for M_1 and M_2.*
(vi) *The closure operator of $M_1 \oplus M_2$ is given by*

$$\mathrm{cl}_{M_1 \oplus M_2}(X) = \mathrm{cl}_{M_1}(X \cap S_1) \cup \mathrm{cl}_{M_2}(X \cap S_2).$$

One way to view the operation of direct sum for graphic matroids is to take the cycle matroid of the union of two graphs that represent the two cycle matroids, using disjoint vertex sets. Note that in general nonisomorphic graphs can have the same cycle matroid. This is illustrated in Figure 8 in which a connected graph and a disconnected graph both have $U_{2,3} \oplus U_{2,3}$ as their direct sum. Whitney showed that if a graph G is 3-connected then G is, up to isomorphism, the only graph with cycle matroid $M(G)$.

This discussion brings up the subject of matroid connectivity, of which we mention just the very beginnings. As the examples in Figure 8 show, the cycle matroid of a graph does not reflect whether the graph is connected, that is, 1-connected. This is to be expected since matroids reflect the structure on the edges of a graph, not the vertices. Thus, we should instead look for connectivity properties that can be expressed solely in terms of edges. An important consequence of Menger's theorem in graph theory is that a graph is 2-connected if and only if each vertex is incident with at least one nonloop and for every pair of nonloop edges e and e' there is a cycle C that contains e and e'. This suggests the next definition.

Definition 2.27. *A matroid M on the ground set S is* connected *if for each pair x and y of elements of S there is a circuit C of M that contains both x and y.*

Thus, a cycle matroid $M(G)$ is connected if and only if the graph G is 2-connected and has no loops (isolated vertices, which affect connectivity in the graph sense, play no role in the cycle matroid).

There are several important reformulations of connectivity in matroids, of which we will use the one in the following theorem. Half of this theorem is very easy to prove; proving the other half requires more effort.

Theorem 2.28. *A matroid is connected if and only if it is not a direct sum of two nonempty matroids.*

Thus, a matroid M that is not connected can be decomposed as a direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ with at least two factors. If, in such a decomposition of M, none of $M_1, M_2, \ldots, M_k$ is empty and all are connected, then the ground sets of $M_1, M_2, \ldots, M_k$ are called the *connected components*, or simply the *components*, of M. It is easy to see that the decomposition of M into components is unique; two elements of M are in the same component if and only if there is a circuit of M that contains both.

There is a theory of higher connectivity for matroids, and this theory is very important in many areas (for example, in the theory of representability of matroids over fields) but we will not get into this deep area. (See Chapter 8 of [26] for a discussion of higher connectivity; note also the major role that 3-connectivity plays throughout much of the second half of that book.) We note one interesting historical item. Unlike graph connectivity, matroid connectivity has the property that a matroid and its dual have the same connectivity. Indeed, W. T. Tutte, who introduced higher connectivity for matroids, *changed* the definition of graph connectivity to match what he introduced for matroids since this property of invariance under duality is so important.

2.3. **Representations of Matroids over Fields.** Recall that a matroid M is representable over a field F if the elements of M can be identified with the columns of some matrix A with entries in F such that a set in M is independent if and only if the corresponding columns of A form a linearly independent *set* of vectors over F. (Note that the same vector can appear in many columns of A but two or more identical columns of A correspond to dependent elements in $M[A]$.)

One fact we will use later is that the Fano plane is representable over a field F if and only if the characteristic of F is 2. Proving this result will give a quick introduction to some useful basic ideas one can apply in the area of representations of matroids.

Theorem 2.29. *The Fano plane is representable over a field F if and only if the characteristic of F is 2.*

Proof. Note that the following operations preserve the independence structure of the columns of a matrix.

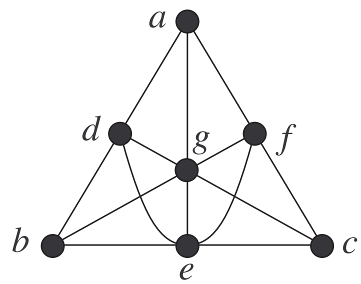

FIGURE 9. The Fano plane, F_7.

(i) Multiply a row by a nonzero element of F.
(ii) Multiply a column by a nonzero element of F.
(iii) Add a scalar multiple of one row to another row.
(iv) Permute the rows.
(v) Permute the columns, along with their labels.
(vi) Delete or add rows of all zeroes.

Thus, these operations can be applied to any matrix that represents a given matroid M over F and the result is another matrix representation of M over F. There is one more operation that can be applied (acting on every entry of A with a fixed automorphism of F) but this will not enter into our work here. (In essence, the operations (i)–(v) above together with acting on the elements of the matrix by an automorphism of the field correspond to semilinear transformations, which are the automorphisms of projective geometries.)

Consider a matrix A that represents F_7, as labelled in Figure 9, over some field F. Note that $\{a, b, c\}$ is a basis. We can apply operations (i)–(vi) to make the columns of A that represent a, b, c be first and to form a 3×3 identity submatrix of A. Furthermore, we can scale the columns for d, e, f, g so that the matrix A has the following form where $*$ denotes an arbitrary nonzero element of F.

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & * & 1 & 0 & * \\ 0 & 0 & 1 & 0 & * & * & * \end{pmatrix}$$

We can scale the second and third rows, and then adjust the second and third columns by scaling, to convert the $*$s in the columns for d and e to 1s, resulting in the following matrix in which we have given names to the remaining unknown entries.

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 0 & 1 & x & z \end{pmatrix}$$

From the dependence of c, d, g, it follows that y is 1. From the dependence of a, e, g, it follows that z and y are the same. Thus, the representing matrix has the

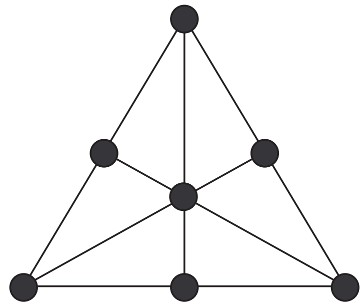

FIGURE 10. The non-Fano matroid, $\bar{F}_7$.

following form.

$$\begin{matrix} a & b & c & d & e & f & g \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & x & 1 \end{pmatrix}$$

From the dependence of d, e, f it follows that x is -1. However, from the dependence of b, f, g it follows that x is 1. Thus, 1 is -1 so F must have characteristic 2. It is also immediate that over any field of characteristic 2 the matrix

$$\begin{matrix} a & b & c & d & e & f & g \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

indeed represents F, thus completing the proof of the theorem. □

Those for whom such arguments are new may want to do the following useful exercises. The non-Fano matroid is the matroid $\bar{F}_7$ of Figure 10. Show that $\bar{F}_7$ is representable over a field F if and only if the characteristic of F is not 2. From this result and Theorem 2.29, deduce that the direct sum $\bar{F}_7 \oplus F_7$ is not representable over any field.

The Fano and non-Fano matroids are two of a family of matroids known as the Reid geometries — after Ralph Reid, who gave the first (never published) proof of the excluded-minor characterization for representability over the field GF(3) — that geometrically encode statements of the type $k = 0$ or $k \neq 0$. The Reid geometries play some role in extremal matroid theory (see [19]), although we will not mention them further in these talks.

We should mention that, contrary to what might be suggested by the proof of Theorem 2.29, in general not all matrix representations of a matroid M over a given field F can be related by the seven operations that are mentioned in that proof. This is due to three facts: matrix representations of geometries are essentially embeddings in projective geometries; the seven operations mentioned in the proof of Theorem 2.29, except operation (vi), correspond to the automorphisms of projective geometries; and some geometries can be embedded in projective geometries in geometrically incompatible ways. (We will see more about projective geometries

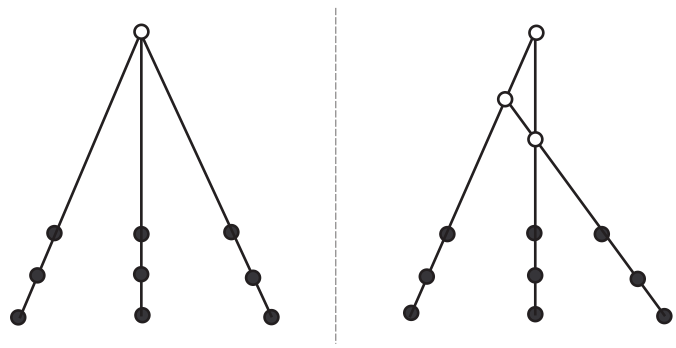

FIGURE 11. One possible cause of inequivalent representations.

soon.) Representations that are not related by the the seven operations in the proof of Theorem 2.29 are called *inequivalent representations*. Without showing the whole of the ambient projective geometry, Figure 11 suggests two embeddings, in a projective plane, that correspond to inequivalent representations of a rank-3 geometry on nine points (the solid dots) that has three disjoint 3-point lines. These embeddings suggest one type of problem that can arise in sufficiently large fields: embeddings may imply that certain points of projective space not actually in the matroid (the hollow dots in the pictures) either are or are not the same.

The existence of inequivalent representations is a key issue that makes the subject of representability for matroids very difficult. To make matters worse, it is known that not only do some matroids have inequivalent representations, but for any prime power q greater than five there is no bound on the number of inequivalent representations that 3-connected matroids can have over GF(q). (See [29].)

Another basic result that will be relevant at several points in these talks is that graphic matroids and cographic matroids are representable over all fields. Since deletions of matroids that are representable over a field F are also representable over F (simply delete appropriate columns from a matrix representation of the larger matroid), and since parallel elements can be realized in matrix representations by repeating columns and loops can be realized by adding columns of zeroes, to show that graphic matroids are representable over all fields it suffices to show that the cycle matroid $M(K_n)$ of the complete graph K_n is representable over all fields. Index the columns of an $n \times \binom{n}{2}$ matrix A_n with the pairs (i, j) with $1 \leq i < j \leq n$, say in lexicographic order. Column (i, j) of A_n has a 1 in row i, a -1 is row j, and zeroes in all other entries. Thus, A_n is simply a signed vertex-edge incidence matrix. (Fields of characteristic 2 will not be an exception; for such fields, the matrix A_n will be a 0-1 matrix.) For instance, for $n = 5$, we get the following matrix.

$$A_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

Identify column (i,j) of A_n with the edge $\{i,j\}$ of K_n. Since the circuits of a matroid are the minimal dependent sets, in order to show that the matroids $M[A_n]$ and $M(K_n)$ are isomorphic it suffices to show that, with this identification, each circuit of $M(K_n)$ is dependent in $M[A_n]$ and that each circuit of $M[A_n]$ is dependent in $M(K_n)$. Let C be a circuit of $M[A_n]$. Note that if the ith entry of some column in C is nonzero, then the ith entries of at least two columns in C are nonzero. It follows that in the subgraph of K_n on the edges corresponding to the columns of C, each vertex of positive degree has degree at least two; from this observation, it is immediate that this set of edges contains a cycle of K_n and so is dependent in $M(K_n)$. Conversely, let C be a circuit in K_n. Observe that by permuting columns, permuting rows, and scaling columns by -1 in a natural manner determined by tracing around the circuit in the graph, it is possible to transform the submatrix of A_n made up of the columns that correspond to edges of C into a matrix of the following form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since the sum of these columns is the zero vector, these columns are dependent. Thus, $M[A_n]$ is a matrix representation of $M(K_n)$.

In the matrix A_n, add all rows, except the first, to the first row of A_n; the resulting first row is a zero row. Let A_n' be the matrix obtained by striking out that zero row and then multiplying the first $n-1$ columns by -1. Note that A_n' is an $n-1$ by $\binom{n}{2}$ matrix whose initial $n-1$ columns form an identity matrix; the other $\binom{n-1}{2}$ columns are all vectors with $n-3$ zeros, a 1 in some row i, and a -1 in some row j with $j > i$. For instance, the matrix obtained in this way from A_5 is the following matrix.

$$A_5' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

Since the operations used to obtain A_n' are among the operations mentioned in the proof of Theorem 2.29, it follows that A_n' is also a matrix representation of $M(K_n)$ over any field F.

Standard arguments about matrices (arguments that also play a role in many other branches of mathematics, such as design theory) show that if a matroid M that is not a free matroid is represented over a field F by an $n \times m$ matrix of the form $[I_n|D]$ where I_n is the $n \times n$ identity matrix, then the dual matroid M^* is represented over F by the matrix $[-D^t|I_{m-n}]$ where D^t is the transpose of D. In particular, the dual of any matroid that is representable over F is also representable over F; thus, the class of matroids that are representable over F is closed under duality. Since we just showed that graphic matroids are representable over all fields, it follows that cographic matroids are representable over all fields.

Theorem 2.30. *Graphic matroids and cographic matroids are representable over all fields.*

We get the following theorem from three results we have mentioned: that duals of matroids that are representable over a field F are also representable over F; that deletions of matroids that are representable over a field F are also representable over F; that deletion and contraction are dual operations.

Theorem 2.31. *For any field F, the class of matroids that are representable over F is closed under minors.*

There are a variety of ways one may attempt to characterize the class of matroids that are representable over a given field F. By far the most successful of these ways is the excluded-minor approach. We will say more about excluded-minor results in later talks. Now we simply mention that there are only three fields F for which the excluded minors for representability over F are known: GF(2), GF(3), and GF(4). This is related to the fact that these fields are sufficiently small that the issue of inequivalent representations does not arise for sufficiently connected (3-connected) matroids. Indeed, over GF(2) and GF(3), no matroid has inequivalent representations.

2.4. **Projective and Affine Geometries.** In a later talk, we will discuss projective and affine geometries in full generality. Here we focus on the projective and affine geometries that will be of most frequent interest to us, namely projective and affine geometries over finite fields.

Definition 2.32. *The* rank-n projective geometry $\mathrm{PG}(n-1,q)$ over the field $\mathrm{GF}(q)$ *is the simplification of the matroid on* $\big(\mathrm{GF}(q)\big)^n$*, the set of n-tuples over* $\mathrm{GF}(q)$*, in which independence is linear independence.*

The $n-1$ in the notation $\mathrm{PG}(n-1,q)$ reflects classical projective geometry, which considers $\mathrm{PG}(n-1,q)$ to have geometric dimension $n-1$. Keep in mind that $\mathrm{PG}(n-1,q)$ has rank n, as is clear from the definition.

Alternatively, $\mathrm{PG}(n-1,q)$ is $M[A]$ where the matrix A over $\mathrm{GF}(q)$ has no zero column and contains, as columns, exactly one vector from each 1-dimensional subspace of $\big(\mathrm{GF}(q)\big)^n$. From this perspective, it is immediate that $\mathrm{PG}(n-1,q)$ is the largest rank-n geometry (simple matroid) that is representable over the field $\mathrm{GF}(q)$ and that any rank-n geometry that is representable over $\mathrm{GF}(q)$ is, up to isomorphism, a restriction of $\mathrm{PG}(n-1,q)$. Also, matrix representations of geometries are essentially embeddings in $\mathrm{PG}(n-1,q)$. Thus, the projective geometries $\mathrm{PG}(n-1,q)$ play the same type of role as "universal models" for geometries that are representable over $\mathrm{GF}(q)$ that the complete graphs K_n play for graphs on n vertices, or that, correspondingly, the cycle matroids $M(K_n)$ play for graphic geometries. We will develop some aspects of this theme much further in later parts of these talks. (Only five types of classes of matroids that are closed under minors and direct sums have such universal models; see [17].)

Since a matroid and its simplification have isomorphic lattices of flats, it follows that the lattice of flats of $\mathrm{PG}(n-1,q)$ is isomorphic to the lattice of subspaces of the vector space $\big(\mathrm{GF}(q)\big)^n$. From this observation we get a number of well-known facts that we will use frequently in parts of these talks and which we briefly mention now.

Since distinct coplanar lines in $\mathrm{PG}(n-1,q)$ are essentially distinct 2-dimensional subspaces in a 3-dimensional subspace of $\big(\mathrm{GF}(q)\big)^n$, and since such subspaces intersect in a 1-dimensional subspace by the dimension theorem given in equation (1), it follows that distinct coplanar lines in $\mathrm{PG}(n-1,q)$ intersect in a point. Thus, projective geometry is quite different from Euclidean geometry.

Upper intervals in the subspace lattice can be viewed as the lattices of subspaces of vector space quotients, and such quotients are, of course, smaller vector spaces. Thus we get the following result.

Theorem 2.33. *If X is a flat of rank k in $\mathrm{PG}(n-1,q)$, then the simplification of the contraction $\mathrm{PG}(n-1,q)/X$ is isomorphic to $\mathrm{PG}(n-k-1,q)$.*

A number of arguments in later talks will use elementary enumerative results about $\mathrm{PG}(n-1,q)$. The most basic of these results is that the number of k-dimensional subspaces of $\mathrm{PG}(n-1,q)$ is given by the Gaussian coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} = \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1}. \tag{2}$$

This formula follows by counting the linearly ordered bases of k-dimensional subspaces of $\big(\mathrm{GF}(q)\big)^n$ (there are $(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$ such linearly ordered bases), then dividing by the number of linearly ordered bases for each k-dimensional subspace (there are $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$ ordered bases for each subspace), and simplifying.

Gaussian coefficients have the following symmetry property, which is easy to check:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$$

As a special case of equation (2), we get that $\mathrm{PG}(n-1,q)$ has $(q^n-1)/(q-1)$ points. Since the rank-$(k+1)$ flats that cover a given rank-k flat X in $\mathrm{PG}(n-1,q)$ can be identified with the points of the contraction $\mathrm{PG}(n-1,q)/X$, which has rank $n-k$, it follows that each rank-k flat in $\mathrm{PG}(n-1,q)$ has $(q^{n-k}-1)/(q-1)$ covers. In particular, each point of $\mathrm{PG}(n-1,q)$ is in $(q^{n-1}-1)/(q-1)$ lines.

Either directly by considering subspaces or via Gaussian coefficients, it follows that all lines in $\mathrm{PG}(n-1,q)$ have $q+1$ points. It follows that the uniform matroid $U_{2,q+1}$ is representable over $\mathrm{GF}(q)$ and that the uniform matroid $U_{2,q+2}$ is one of the excluded minors for representability over $\mathrm{GF}(q)$.

By using linear transformations, one can easily show that the automorphism group of $\mathrm{PG}(n-1,q)$ is transitive on the collection of flats of any fixed rank. In particular, for all hyperplanes H of $\mathrm{PG}(n-1,q)$, the deletions $\mathrm{PG}(n-1,q)\backslash H$ are isomorphic. Thus, the following definition makes sense.

Definition 2.34. *The* rank-n affine geometry $\mathrm{AG}(n-1,q)$ over the field $\mathrm{GF}(q)$ *is the deletion* $\mathrm{PG}(n-1,q)\backslash H$ *of* $\mathrm{PG}(n-1,q)$ *where H is a hyperplane of* $\mathrm{PG}(n-1,q)$.

It follows that the number of points in $\mathrm{AG}(n-1,q)$ is

$$\frac{q^n-1}{q-1} - \frac{q^{n-1}-1}{q-1},$$

that is, q^{n-1}.

Take a matrix representation $M[A]$ of $\mathrm{PG}(n-1,q)$ in which each column begins with either 0 or 1; take the hyperplane H to be the set of columns whose first entry is 0; from this, it is easy to see that dependence in the affine geometry $\mathrm{AG}(n-1,q) = \mathrm{PG}(n-1,q)\backslash H$ is precisely affine dependence in $\big(\mathrm{GF}(q)\big)^{n-1}$. (Recall that vectors $v_1, v_2, \ldots, v_k$ are *affinely dependent over* $\mathrm{GF}(q)$ if there are elements $\alpha_1, \alpha_2, \ldots, \alpha_k$ in $\mathrm{GF}(q)$, not all zero, with $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 0$.)

By the dimension theorem given in equation (1), each line of $\mathrm{PG}(n-1,q)$ that is not in a hyperplane H intersects H in exactly one point. Thus, since all lines of $\mathrm{PG}(n-1,q)$ have exactly $q+1$ points, it follows that all lines of $\mathrm{AG}(n-1,q)$ have exactly q points. Coplanar lines of $\mathrm{PG}(n-1,q)$ that are not in H but that intersect in a point of H give rise to coplanar, disjoint lines in $\mathrm{AG}(n-1,q)$, that is, parallel lines of $\mathrm{AG}(n-1,q)$.

Let x be a point of $\mathrm{AG}(n-1,q)$. Since $\mathrm{AG}(n-1,q)$ has q^{n-1} points and each line through x has $q-1$ points in addition to x, it follows that there are $(q^{n-1}-1)/(q-1)$ lines of $\mathrm{AG}(n-1,q)$ through x. Thus, there are $(q^{n-1}-1)/(q-1)$ points in the rank-$(n-1)$ geometry $\mathrm{si}\big(\mathrm{AG}(n-1,q)/x\big)$. Since this is the maximal number of points in a rank-$(n-1)$ geometry that is representable over $\mathrm{GF}(q)$, it follows that $\mathrm{si}\big(\mathrm{AG}(n-1,q)/x\big)$ is isomorphic to $\mathrm{PG}(n-2,q)$. Thus, affine geometries are deletions of projective geometries and projective geometries are simplifications of contractions of affine geometries.

3. A First Taste of Extremal Matroid Theory: Cographic Matroids

Broadly stated, extremal matroid theory is concerned with the relations between various parameters of a matroid. Many, many theorems can be described as being of this type. We will focus on the following more specific aspect: determine the greatest number of elements that a matroid can have, usually as a function of rank, given that the matroid satisfies certain side conditions. Ideally, we would also like to classify all examples that have the maximal number of elements, that is, all examples that show that the upper bound on the number of elements cannot be improved (if indeed it cannot be improved). We will start with two trivial examples followed by one with more substance; these examples will be a useful first exposure to illustrate the type of questions that are of interest.

If the restriction of interest is that the matroid must be simple and representable over $\mathrm{GF}(q)$, then clearly $(q^n-1)/(q-1)$ is an upper bound on the number of elements among such matroids of rank n. Clearly also there is only one matroid of rank n that meets the bound, namely $\mathrm{PG}(n-1,q)$.

If the restriction of interest is that the matroid must be simple and graphic, then clearly $\binom{n+1}{2}$ is an upper bound on the number of elements among such matroids of rank n. Clearly also there is only one matroid of rank n that meets the bound, namely $M(K_{n+1})$.

What makes the observations in the last two paragraphs so transparent is that in each case there is an obvious maximal geometry of each rank in which all of the geometries of interest of the corresponding rank can be embedded. Every rank-n geometry that is representable over $\mathrm{GF}(q)$ can be embedded in $\mathrm{PG}(n-1,q)$; every rank-n graphic geometry can be embedded in $M(K_{n+1})$. There are other cases we will see later in which we will have such maximal geometries; in other cases we will consider there will be no such maximal geometries. The main point of this part

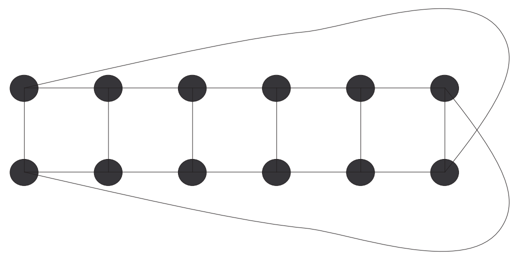

FIGURE 12. The "Möbius ladder" G_7.

of the talks is to introduce extremal matroid theory through a simple example of this latter type. The main theorem of this section addresses the following natural question: *What is the maximum number of points in a simple cographic matroid of rank n?*

Theorem 3.1. *Let M be a cographic geometry of rank n with n at least 2. Then M has at most $3n-3$ elements. Furthermore, cographic geometries of rank n with $3n-3$ elements are connected and are the cocycle matroids of connected trivalent graphs with $2n-2$ vertices in which edge-cutsets have at least three edges.*

It is worth noting how different the classes of graphic and cographic geometries are: for graphic geometries the bound on the number of elements is quadratic in the rank while for cographic geometries the bound is linear in the rank; among graphic geometries there is a unique geometry of each rank that meets the bound while among cographic geometries there are many such geometries. Also, as we will see, the proof of Theorem 3.1, while still fairly easy, is not nearly as simple as the corresponding result for graphic geometries.

We first show, by exhibiting an example, that cographic geometries of rank n can have at least $3n-3$ elements. Consider the graph G_n, sometimes called the "Möbius ladder", that is formed as follows. The vertex set consists of integer points in the plane of the forms $(i,0)$ and $(i,1)$ for i with $0 \leq i \leq n-2$. The edges connect points that are distance 1 from each other; also, there is an edge connecting $(0,1)$ and $(n-2,0)$, and an edge connecting $(0,0)$ and $(n-2,1)$. (The "twist" in these last two edges plays a significant role only in the cases $n=2$ and $n=3$; note that G_2 has three parallel edges.) The "Möbius ladder" G_7 is shown in Figure 12. Note that the smallest edge-cutsets in G_n have three edges; thus, the cocycle matroid $M^*(G_n)$ is simple. Since G_n has $2n-2$ vertices, each of degree 3, the number of edges is $3(2n-2)/2$, which is $3n-3$. Since G_n is a connected graph with $2n-2$ vertices, the cycle matroid $M(G_n)$ has rank $2n-3$. Therefore by Observation 2.8 it follows that the cocycle matroid $M^*(G_n)$ has rank n. Thus, there are cographic geometries of rank n with $3n-3$ elements.

We next show that cographic geometries of rank n have no more than $3n-3$ elements. Note that the number of elements in any matroid is the sum of the numbers of elements in its connected components; also, the rank of a matroid is the sum of the ranks of its connected components. From this it is clear that it suffices to prove the bound for *connected* cographic geometries and that, once we

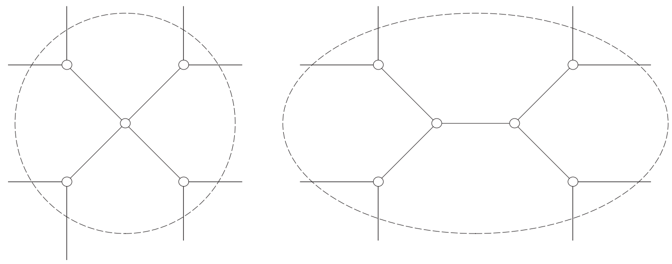

FIGURE 13. "Splitting" a vertex to reduce the degree.

have shown this it follows that disconnected cographic geometries of rank n have strictly fewer than $3n-3$ elements.

Note that in any graph G for which the cocycle matroid $M^*(G)$ is to be a connected geometry, there can be no edge-cutsets consisting of one or two edges since such edge-cutsets would give rise to loops or parallel elements in $M^*(G)$. Thus, in such a graph all vertices must have degree at least three. Also, such a graph is necessarily connected. The converse, however, is not true: for instance, the graph that consists of a 4-cycle with parallel edges added to two nonadjacent edges is connected and has all vertices of degree three yet there is a pair of edges that forms a cutset, so the corresponding cocycle matroid is not a geometry. Let $\mathcal{T}$ denote the class of cocycle matroids that can be represented in the form $M^*(G)$ where G is a connected graph in which all vertices have degree three or more. Since all connected cographic geometries are in $\mathcal{T}$, to prove our desired upper bound, it suffices to show that rank-n matroids in $\mathcal{T}$ have at most $3n-3$ elements.

Thus, assume that G is a connected graph in which all vertices have degree at least three. As suggested in Figure 13, if G has a vertex of degree exceeding three, we can "split" that vertex and obtain another (typically not unique) connected graph, say G', with one more edge, say e', having the properties that all degrees in G' are at least three and G is obtained from G' by contracting e'. Thus, $M^*(G')$ is in the class $\mathcal{T}$. It is clear that the edge e' of G' can be added to any spanning tree of G to get a spanning tree of G'. Thus, we have the equality $r\big(M(G')\big) = r\big(M(G)\big)+1$. By this equality, Observation 2.8, and the fact that $M(G')$ has one more element than $M(G)$, it follows that $M^*(G')$ and $M^*(G)$ have the same rank. From this, we deduce that among rank-n matroids in $\mathcal{T}$, those with the greatest number of elements arise from graphs in which all degrees are three.

We now provide the final step in the proof of Theorem 3.1, namely, that the number of elements in a rank-n cocycle matroid $M^*(G)$ where G is connected and trivalent (i.e., all vertices have degree 3) must be $3n-3$. Let ϵ be the number of edges of G, that is, the number of elements of $M^*(G)$, and let ν be the number of vertices of G. Since all vertices have degree 3, we have $2\epsilon = 3\nu$, so $\nu = 2\epsilon/3$. Since G is connected, the rank of $M(G)$ is $\nu - 1$. The sum $r\big(M(G)\big) + r\big(M^*(G)\big)$ is the number of elements in the ground sets of these matroids, so we get $\nu - 1 + n = \epsilon$. Thus, $2\epsilon/3 - 1 + n = \epsilon$, so $n - 1 = \epsilon/3$. Thus ϵ, the number of elements in M, is $3n-3$.

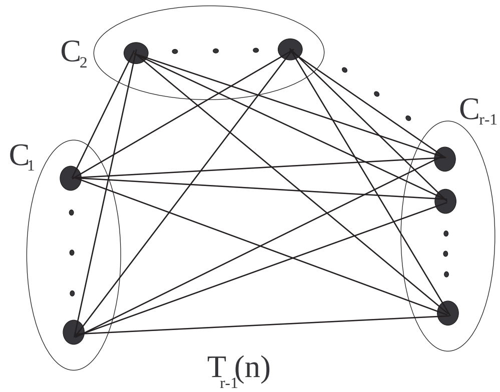

FIGURE 14. A sketch of a Turán graph.

4. Excluding Subgeometries: The Bose-Burton Theorem

Our first extensive topic in extremal matroid theory is the Bose-Burton theorem, a collection of results in the spirit of this theorem, and several related conjectures. We will introduce the Bose-Burton theorem as a counterpart, in projective geometry, of Turán's theorem in extremal graph theory.

To state Turán's theorem, which is commonly regarded as the starting point of much of extremal graph theory, we need the following terminology and notation. The *Turán graph* $T_{r-1}(n)$ is the complete $(r-1)$-partite graph on n vertices in which the $r-1$ classes each have $\lfloor n/(r-1)\rfloor$ or $\lceil n/(r-1)\rceil$ vertices. Thus, the n vertices in the Turán graph $T_{r-1}(n)$ are partitioned into $r-1$ classes; each class has cardinality $\lfloor n/(r-1)\rfloor$ or $\lceil n/(r-1)\rceil$, and how many classes there are of each cardinality depends upon the remainder of n upon division by $r-1$; two vertices are adjacent if and only if they are in different classes. Figure 14 sketches a generic Turán graph. The number of edges in $T_{r-1}(n)$ is the *Turán number* $t_{r-1}(n)$. Turán graphs are of interest since they are the unique graphs that attain the upper bound in the following theorem from extremal graph theory.

Theorem 4.1 (Turán, 1941)**.** *The largest number of edges in a graph on* n *vertices that does not contain* K_r *as a subgraph is* $t_{r-1}(n)$*. Furthermore,* $T_{r-1}(n)$ *is the only such graph having* $t_{r-1}(n)$ *edges.*

In this theorem "graph" refers to "simple graph" and this will be the convention throughout this section; without this condition there is clearly no upper bound on the number of edges. Note that the Turán graph $T_{r-1}(n)$ does not have K_r as a subgraph since among any r vertices at least two are in the same class and so are not adjacent. The other assertions in Turán's theorem are less transparent. There are a wide variety of proofs of Turán's theorem, all of which are fairly easy.

Simple graphs on n vertices can be viewed as subgraphs of K_n. Thus, Turán's theorem is about subgraphs of K_n that do not contain K_r as a subgraph. In light of the remarks above that projective geometries and the cycle matroids of complete graphs both play the role of universal models for certain classes of geometries, it is natural to look for a counterpart of Turán's theorem that involves projective geometries. This is precisely the problem that the Bose-Burton theorem addresses.

Theorem 4.2 (Bose and Burton, 1966)**.** *Let M be $\mathrm{PG}(n-1,q)|S$ (to focus on what remains) or, equivalently, $\mathrm{PG}(n-1,q)\backslash T$ (to focus on what is removed). Assume that M has no subgeometry isomorphic to $\mathrm{PG}(m-1,q)$. Then*

$$|S| \leq \frac{q^n - q^{n-m+1}}{q-1}.$$

Furthermore, $|S|$ is $(q^n - q^{n-m+1})/(q-1)$ if and only if $\mathrm{PG}(n-1,q)|T$ is isomorphic to $\mathrm{PG}(n-m,q)$.

To illustrate this theorem before we turn to the proof, we cite a special case that may be easier to grasp on a first encounter.

Corollary 4.3. *A subgeometry of $\mathrm{PG}(n-1,q)$ that has no lines with $q+1$ points (that is, no $\mathrm{PG}(1,q)$-subgeometry) has at most q^{n-1} points. Furthermore, the only such geometry with q^{n-1} points is the affine geometry $\mathrm{AG}(n-1,q)$.*

Recall that the affine geometry $\mathrm{AG}(n-1,q)$ is constructed by removing a hyperplane H from $\mathrm{PG}(n-1,q)$; also, $\mathrm{PG}(n-1,q)|H$ is isomorphic to $\mathrm{PG}(n-2,q)$. By the dimension theorem given in equation (1), the hyperplane H intersects every line of $\mathrm{PG}(n-1,q)$ in at least one point. Thus, in $\mathrm{PG}(n-1,q)\backslash H$ at least one point has been removed from each line, so lines in $\mathrm{AG}(n-1,q)$ have at most q points. The corollary says that this construction yields the optimal number of points among geometries with at most q points on each line and that this construction is the only way to achieve this optimal number of points.

The key to the proof of the Bose-Burton theorem is basic counting combined with property (iii) of Theorem 2.12. An alternative formulation of property (iii) of Theorem 2.12 is that if $X_1, X_2, \ldots, X_k$ are the covers a flat X of a matroid M, then the set differences $X_1 - X, X_2 - X, \ldots, X_k - X$ partition the elements of M that are not in X.

Proof of the Bose-Burton theorem. We first show that the claimed extremal examples indeed have no $\mathrm{PG}(m-1,q)$-subgeometries, thus showing that subgeometries of $\mathrm{PG}(n-1,q)$ that have no $\mathrm{PG}(m-1,q)$-subgeometries can have at least $(q^n - q^{n-m+1})/(q-1)$ elements. Thus, assume that $\mathrm{PG}(n-1,q)|T$ is isomorphic to $\mathrm{PG}(n-m,q)$; thus, T is a flat of $\mathrm{PG}(n-1,q)$ of rank $n-m+1$. Let F be any flat of $\mathrm{PG}(n-1,q)$ of rank m. We need to show that F is not a flat of $\mathrm{PG}(n-1,q)\backslash T$, that is, that $F \cap T$ is nonempty. This follows immediately from the basic dimension theorem of linear algebra:

$$\underbrace{\dim(F)}_{m} + \underbrace{\dim(T)}_{n-m+1} = \underbrace{\dim(F \cup T)}_{\leq n} + \dim(F \cap T).$$

It follows that $\dim(F \cap T)$ is at least 1, so $F \cap T$ is nonempty, as needed.

We next show that the number of elements in any subgeometry M of $\mathrm{PG}(n-1,q)$ that has no $\mathrm{PG}(m-1,q)$-subgeometry is at most $(q^n - q^{n-m+1})/(q-1)$. The proof is by induction on m. The case $m = 1$ is trivial since the restriction is that M has no points. Thus, assume that the claimed bound holds for subgeometries of $\mathrm{PG}(n-1,q)$ that have no $\mathrm{PG}(m-2,q)$-subgeometry and consider a restriction $M := \mathrm{PG}(n-1,q)|S$ of $\mathrm{PG}(n-1,q)$ that has no $\mathrm{PG}(m-1,q)$-subgeometry. If M has no $\mathrm{PG}(m-2,q)$-subgeometry, then by the induction hypothesis, we get

$$|S| \leq \frac{q^n - q^{n-m+2}}{q-1},$$

so $|S|$ is strictly less than $(q^n - q^{n-m+1})/(q-1)$. Thus, we can assume that M has a PG$(m-2, q)$-subgeometry; say the restriction of M to the flat F is isomorphic to PG$(m-2, q)$. Since $M|F$ is isomorphic to PG$(m-2, q)$, the flat F has $(q^{m-1}-1)/(q-1)$ elements. Since F has rank $m-1$ in a projective geometry of rank n, the simplification of the contraction PG$(n-1, q)/F$ has rank $n-m+1$ and so is isomorphic to PG$(n-m, q)$. In particular, the contraction PG$(n-1, q)/F$ has $(q^{n-m+1}-1)/(q-1)$ points. Thus, F has $(q^{n-m+1}-1)/(q-1)$ covers in the projective geometry PG$(n-1, q)$; therefore F has *at most* $(q^{n-m+1}-1)/(q-1)$ covers in M. (With an eye toward determining the geometries that have the maximal number of elements, observe in what follows that to achieve the maximal number of elements, F must have exactly $(q^{n-m+1}-1)/(q-1)$ covers in M.) Now each cover F' of F in M has rank m, but $M|F'$ cannot be PG$(m-1, q)$. Thus, each cover F' of F in M has at most $(q^m-1)/(q-1)-1$ elements, and so at most

$$\frac{q^m-1}{q-1} - \frac{q^{m-1}-1}{q-1} - 1,$$

or $q^{m-1}-1$, elements in $F'-F$. (Again with an eye toward determining the geometries that have the maximal number of elements, observe in what follows that all such flats F' necessarily have exactly $q^{m-1}-1$ elements in $F'-F$ if the maximum number of elements is to be achieved in M.) Thus, from the size of F, the upper bound on its number of covers, and the upper bound on the number of elements each of these covers can have outside of F, we get this inequality:

$$\begin{aligned} |S| &\le |F| + \frac{q^{n-m+1}-1}{q-1}(q^{m-1}-1) \\ &= \frac{q^{m-1}-1}{q-1} + \frac{q^{n-m+1}-1}{q-1}(q^{m-1}-1) \\ &= \frac{q^{m-1}-1}{q-1}(1+q^{n-m+1}-1) \\ &= \frac{q^n-q^{n-m+1}}{q-1}, \end{aligned}$$

which is the inequality that we needed to show.

Lastly we examine the geometries that attain the bound. Assume that M is the restriction PG$(n-1, q)|S$ of PG$(n-1, q)$, that $|S| = (q^n - q^{n-m+1})/(q-1)$, and that M has no PG$(m-1, q)$-subgeometry. By the observations above, any flat Z of M with $M|Z$ isomorphic to PG$(m-2, q)$ has exactly $(q^{n-m+1}-1)/(q-1)$ covers in M and each of these covers has exactly $q^{m-1}-1$ elements that are not in Z. Let T be the set of elements of PG$(n-1, q)$ that are not in M. Thus,

$$|T| = \frac{q^n-1}{q-1} - \frac{q^n-q^{n-m+1}}{q-1} = \frac{q^{n-m+1}-1}{q-1}.$$

The theorem will be proven by showing that T is a flat. Now the flats of PG$(n-1, q)$ are characterized by a simple statement that is the geometric counterpart of the definition of a subspace of a vector space: a set X in PG$(n-1, q)$ is a flat if and only if whenever x and y are elements of X, the line cl$(\{x, y\})$ is contained in X. So assume x and y are in T and z is in cl$(\{x, y\})$ but that z is not in T. We will derive a contradiction. Thus, we are assuming that z is in S. Now z is contained in $(q^{n-1}-1)/(q-1)$ lines of PG$(n-1, q)$. Note that the number of covers of z in PG$(n-1, q)$ exceeds the number of points in T if and only if $n-1 > n-m+1$, that is, if and only if $m > 2$; thus, if m exceeds 2, then, by the size of T, at least one of the lines of $\mathrm{PG}(n-1,q)$ that contain z, say Z_2, is entirely contained in M. Now Z_2 is contained in $(q^{n-2}-1)/(q-1)$ planes of $\mathrm{PG}(n-1,q)$. Note that the number of covers of Z_2 in $\mathrm{PG}(n-1,q)$ exceeds the number of points in T if and only if $n-2 > n-m+1$, that is, if and only if $m > 3$; thus, if m exceeds 3, then, by the size of T, at least one of the planes that covers Z_2 in $\mathrm{PG}(n-1,q)$ is entirely contained in M. In this way it follows that there is a flat Z of M that contains z with $M|Z$ isomorphic to $\mathrm{PG}(m-2,q)$. Since x and y are collinear with z, they are in the same cover, say Z', of Z in $\mathrm{PG}(n-1,q)$. Now we have one of two contradictions: either Z' does not give rise to a cover of Z in M, so Z has fewer than $(q^{n-m+1}-1)/(q-1)$ covers in M, or Z' does give rise to a cover of Z in M, but this cover has fewer than $q^{m-1}-1$ points (since both x and y are missing). These contradictions show that z must have been in T, thus T is a flat of $\mathrm{PG}(n-1,q)$, and so $\mathrm{PG}(n-1,q)|T$ is isomorphic to $\mathrm{PG}(n-m,q)$, as claimed. □

Just as Turán's theorem is of central importance in extremal graph theory, so the Bose-Burton theorem and the geometries that arise in this theorem appear to play a key role in extremal matroid theory, or, more precisely, the part of extremal matroid theory that will be our focus in this section of these talks. In many problems in extremal matroid theory we impose some side conditions on a matroid and we ask for an upper bound on the number of elements, perhaps as a function of the rank. In the problems we consider in this part of these talks, the restrictions we impose are on submatroids; in later sections, we impose restrictions on minors. Problems involving excluded submatroids form a much underdeveloped area of extremal matroid theory and one in which the Bose-Burton theorem seems to be of prime importance. Making this precise is still conjectural, but we will sample some of the evidence and see what the major conjectures are.

We first examine the geometry that arises in the Bose-Burton theorem. Since, up to isomorphism, all that matters in this geometry is the rank of the ambient projective geometry and that of the flat that is deleted, we can use the notation $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$ to denote this geometry. This geometry is clearly the largest subgeometry of $\mathrm{PG}(n-1,q)$ that is disjoint from some flat of $\mathrm{PG}(n-1,q)$ of rank $n-m+1$. This may sound like a minor point, but this concept is of central importance in a branch of matroid theory called *the critical problem*. We will start with a definition and then suggest why this concept appears to be relevant to extremal matroid theory.

Definition 4.4. *Let M be a geometry that can be embedded in* $\mathrm{PG}(n-1,q)$*. The* critical exponent of M over $\mathrm{GF}(q)$*, denoted $c(M;q)$, is the least positive integer k such that the image of M in* $\mathrm{PG}(n-1,q)$ *under an embedding is disjoint from some flat of* $\mathrm{PG}(n-1,q)$ *of rank $n-k$, that is, corank k.*

Thus, the critical exponent of M over $\mathrm{GF}(q)$ is the least positive integer k for which M can be embedded in the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-1-k,q)$. From this perspective, the following lemma is obvious.

Lemma 4.5. *Assume that the geometry M is representable over* $\mathrm{GF}(q)$ *and that M' is a subgeometry of M. Then* $c(M';q) \leq c(M;q)$

Note that the critical exponent of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$ is $m-1$. In particular, the affine geometry $\mathrm{AG}(n-1,q)$, which is $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-2,q)$, has critical exponent 1 and is the largest rank-n geometry that is representable over $\mathrm{GF}(q)$ and has critical exponent 1 over $\mathrm{GF}(q)$. In general, we call a geometry that is representable over $\mathrm{GF}(q)$ *affine* if it has critical exponent 1 over $\mathrm{GF}(q)$; thus, the affine geometries over $\mathrm{GF}(q)$ are the subgeometries of the geometries $\mathrm{AG}(n-1,q)$. (Affine geometries are also the geometries in which independence can be interpreted as affine independence as described after Definition 2.34.)

The notation $c(M;q)$ for the critical exponent reflects the field $\mathrm{GF}(q)$; this is important since a matroid may be representable over many fields and its critical exponents typically differ over different fields.

Observe that the notation $c(M;q)$ does not reflect the embedding or even the rank n of the projective geometry into which M is being embedded. It is a theorem, due to Henry Crapo and Gian-Carlo Rota that the critical exponent is independent of both the embedding and the dimension in which the embedding is done. A bit more precisely, what Crapo and Rota showed is that the critical exponent is the least positive integer k such that $\chi(M;q^k)$ is nonzero, where $\chi(M;x)$ is a certain polynomial invariant one can associate with a matroid; this explains the terminology critical *exponent*, and it also shows that the invariant depends only on the matroid and the order q of the field, not on the embedding or the dimension. The polynomial $\chi(M;x)$, known as the characteristic polynomial of the matroid, can be viewed as coming from Möbius inversion, which is crucial in the proof of the Crapo-Rota theorem, or as a special evaluation of the Tutte polynomial of M. However, explaining these results as fully as they deserve would take us too far afield. Indeed, the critical problem is a rather deep and difficult area of matroid theory, and it would take at least one or two talks to explain the basics. A few general comments will have to suffice in this context. The thrust of the critical problem is to provide a unified framework for studying a large number of problems in combinatorics, including graph coloring and the fundamental problem of linear coding theory. The critical number gives, in a sense that can be made precise, a counterpart of the chromatic number of a graph. We refer to [20] for a state-of-the-art survey of the critical problem. We will focus on why the critical exponent appears relevant to the part of extremal matroid theory that we are considering now.

We start the explanation of why the critical exponent seems to be relevant to extremal matroid theory by returning to extremal graph theory. Turán's theorem has many deep extensions, of which we mention only two. First, we need some notation. For a graph H, let $ex(H;n)$ be

$$ex(H;n) := \max\{|E(G)| \,:\, |V(G)| = n \text{ and } H \not\leq G\},$$

the maximum number of edges among (simple) graphs on n vertices that do not have H as a subgraph. For example, by Turán's theorem we have $ex(K_r;n) = t_{r-1}(n)$. In general, it is extremely difficult to find an expression for the function $ex(H;n)$. Thus, the best hope is for an asymptotic estimate for $ex(H;n)$. The following beautiful theorem gives such an estimate.

Theorem 4.6 (Erdös-Simonovits, 1966)**.** *Let $\chi(H)$ denote the chromatic number of a graph H that has at least one edge. Then*

$$\lim_{n\to\infty} \frac{ex(H;n)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H)-1}.$$

The chromatic number $\chi(H)$ of a graph H is the least positive integer k such that there is a function (a coloring) from the set of vertices of H into the set $\{1, 2, \ldots, k\}$ (which one can think of as a set of k colors) such that adjacent vertices have distinct images. Equivalently, the chromatic number of H is the least positive integer k for which the vertices of H can be partitioned into k classes so that no vertices in the same class are adjacent. This perspective makes it clear that the chromatic number of H is the least positive integer k such that H is a subgraph of a Turán graph $T_k(n)$ for sufficiently large n. This formulation of the chromatic number is a counterpart of the formulation of the critical exponent in terms of the geometries $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-1-k,q)$ given right after Definition 4.4. The chromatic number $\chi(H)$ is also the least positive integer k such that the chromatic polynomial $\chi(G;x)$ is nonzero at k. The characteristic polynomial of a matroid generalizes the chromatic polynomial of a graph. One should note the similarity between the last formulation of the chromatic number and the theorem of Crapo and Rota that we mentioned above.

Note that the ratio $ex(H;n)/\binom{n}{2}$ in the Erdös-Simonovits theorem is the ratio of the maximum number of edges that graphs on n vertices may have if H is not a subgraph to the maximum number of edges that graphs on n vertices may have with no restriction. One of the things that is so beautiful about Theorem 4.6 is that it relates two parameters, $ex(H;n)$ and $\chi(H)$, of the graph H in a very simple, powerful, and striking way.

It in natural to ask if there is a counterpart of the Erdös-Simonovits theorem for representable matroids. This is currently unknown, but there is a conjecture that is suggested by the Bose-Burton theorem. To make it easier to state this conjecture and the supporting evidence, we first adapt the notation of extremal graph theory as follows. For a geometry M that is representable over $\mathrm{GF}(q)$, let $ex_q(M;n)$ be defined as follows:

$$ex_q(M;n) = \max\{|S| : \mathrm{PG}(n-1,q)|S \text{ does not have } M \text{ as a subgeometry}\}.$$

Thus, the Bose-Burton theorem gives

$$ex_q\big(\mathrm{PG}(m-1,q);n\big) = \frac{q^n - q^{n-m+1}}{q-1}.$$

A counterpart of the Erdös-Simonovits theorem would have to agree with the following calculation:

$$\lim_{n\to\infty} \frac{ex_q\big(\mathrm{PG}(m-1,q);n\big)}{\frac{q^n-1}{q-1}} = \lim_{n\to\infty} \frac{\frac{q^n - q^{n-m+1}}{q-1}}{\frac{q^n-1}{q-1}} = 1 - \frac{1}{q^{m-1}}.$$

Note that the exponent $m-1$ that appears in the last expression is one less than the critical exponent of $\mathrm{PG}(m-1,q)$, which, as we have suggested, is the counterpart, for a representable matroid, of the chromatic number of a graph.

This suggests the following conjecture.

Conjecture 4.7. *Let c be the critical exponent over* $\mathrm{GF}(q)$ *of a geometry M that is representable over* $\mathrm{GF}(q)$. *Then*

$$\lim_{n\to\infty} \frac{ex_q(M;n)}{\frac{q^n-1}{q-1}} = 1 - \frac{1}{q^{c-1}}.$$

This conjecture is implicit in the excellent survey of extremal matroid theory by Joseph Kung [19] but it has never been explicitly stated in print. Still, it has been the (unstated) motivation behind some recent results in extremal matroid theory, primarily those that appear in [10]. The point of the next several topics we will cover is to provide some evidence for this conjecture and to prove one special case; in the process, we will see more about the central role that the geometries $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$ play in this area of extremal matroid theory. Before doing this, though, we should mention another result from graph theory and its conjectured matroid counterpart. This conjecture too has never appeared in print.

The following theorem of Erdös and Stone is reasonably difficult to prove; there are a variety of proofs of the result (see, e.g., [11, 22]) but all of the proofs are fairly involved. However, the Erdös-Simonovits theorem follows from the Erdös-Stone theorem as a fairly easy corollary. (It would be nice to have a proof of the Erdös-Simonovits theorem that does not use the Erdös-Stone theorem and that is considerably simpler than the present proofs of the Erdös-Stone theorem; I am unaware of such a proof.)

Theorem 4.8 (Erdös-Stone, 1946)**.** *Let r and s be integers with $r \geq 2$ and $s \geq 1$. Let ϵ be positive. There is a positive integer n_0 such that if $n \geq n_0$ and G is a graph with n vertices and at least $t_{r-1}(n) + \epsilon\binom{n}{2}$ edges, then G has $T_r(rs)$ as a subgraph.*

This theorem points out one of the many ways in which Turán graphs play a central role in extremal graph theory: by having the ratio of the number of edges over $\binom{n}{2}$ go just ϵ above $t_{r-1}(n)/\binom{n}{2}$, we get not just one K_r-subgraph but many K_r-subgraphs — how many can be controlled by the choice of s — there are s^r subgraphs isomorphic to K_r in the Turán graph $T_r(rs)$. Of course, this is for large enough n.

A potential counterpart of the Erdös-Stone theorem is given in the following conjecture. Some easy special cases of this conjecture have been verified, but such cases are very limited.

Conjecture 4.9. *Let m and t be integers with $t > m \geq 1$ and let ϵ be positive. There is a positive integer n_0 such that if $n \geq n_0$ and M is a subgeometry of $\mathrm{PG}(n-1,q)$ with at least*

$$\frac{q^n - q^{n-m+1}}{q-1} + \epsilon\frac{q^n-1}{q-1}$$

elements, then M has a subgeometry isomorphic to $\mathrm{PG}(t-1,q)\backslash\mathrm{PG}(t-m-1,q)$.

Analogous to the graph $T_r(rs)$ in the Edrös-Stone theorem (a graph with s^r subgraphs isomorphic to K_r), the geometry $\mathrm{PG}(t-1,q)\backslash\mathrm{PG}(t-m-1,q)$ has $q^{m(t-m)}$ subgeometries isomorphic to $\mathrm{PG}(m-1,q)$. (One can show this using the same technique we used in Section 2.4 to count the number of rank-k flats in $\mathrm{PG}(n-1,q)$.)

Just as the Erdös-Simonovits theorem follows from the Erdös-Stone theorem with only modest work, so the conjectured counterpart of Erdös-Simonovits theorem would follow easily from the conjectured counterpart of Erdös-Stone theorem, as we show next.

Theorem 4.10. *If the counterpart of Erdös-Stone theorem is true, then the counterpart of Erdös-Simonovits theorem is true.*

Proof. Assume that c is the critical exponent over $\mathrm{GF}(q)$ of a geometry M that is representable over $\mathrm{GF}(q)$. By the definition of the critical exponent, it follows that M is a subgeometry of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-c-1,q)$ for every n with $n \geq r(M)$, but M is not a subgeometry of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-c,q)$ for any n. The second observation gives the inequality

$$\frac{q^n - q^{n-c+1}}{q-1} \leq ex_q(M;n)$$

for all n since we have an example of a subgeometry of $\mathrm{PG}(n-1,q)$ that does not have M as a subgeometry and that has $(q^n - q^{n-c+1})/(q-1)$ points. By the first observation, for any n with $n \geq r(M)$, any geometry with no M-subgeometry has no $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-c-1,q)$-subgeometry. Therefore if the counterpart of the Erdös-Stone theorem is valid, this implies that $ex_q(M;n)$ must be less than the bound in the Erdös-Stone theorem, specifically, for any fixed positive number ϵ, we have

$$ex_q(M;n) < \frac{q^n - q^{n-c+1}}{q-1} + \epsilon\frac{q^n-1}{q-1}$$

for sufficiently large n. By combining these two inequalities and dividing by the number of points in $\mathrm{PG}(n-1,q)$, we get

$$\frac{q^n - q^{n-c+1}}{q^n-1} \leq \frac{ex_q(M;n)}{\frac{q^n-1}{q-1}} < \frac{q^n - q^{n-c+1}}{q^n-1} + \epsilon.$$

From this, we get the limit

$$\lim_{n\to\infty} \frac{ex_q(G;n)}{\frac{q^n-1}{q-1}} = 1 - \frac{1}{q^{c-1}}$$

asserted in the conjectured counterpart of the Erdös-Simonovits theorem. □

We have suggested that the geometries $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$, which we will call the Bose-Burton geometries, play a central role in the area of extremal matroid theory that we are considering in this part of these talks. The conjectures just mentioned suggest this, but we should also support this assertion with some theorems. We will start with a result from extremal graph theory, for which we will need a mild extension of the notation $ex(H;n)$. For a set $\mathcal{H}$ of graphs, let

$$ex(\mathcal{H};n) := \max\{|E(G)| : |V(G)| = n \text{ and } H \not\leq G \text{ for all } H \in \mathcal{H}\}.$$

Thus, rather than excluding a single graph H as a subgraph, we are excluding a family $\mathcal{H}$ of graphs as subgraphs. The following theorem appears in [5] without attribution (so it is probably due to Bollobás).

Theorem 4.11. *Let $\mathcal{H}$ be a set of graphs, each of which has at most k vertices and none of which is s-partite. Assume there is an integer n_0 with $n_0 \geq k$ such that the following conditions hold:*

(1) *$ex(\mathcal{H};n_0)$ is the Turán number $t_s(n_0)$, and*
(2) *the only graph on n_0 vertices with $t_s(n_0)$ edges and with no subgraphs in $\mathcal{H}$ is the Turán graph $T_s(n_0)$.*

Then for any n with $n \geq n_0$, we have

(1′) *$ex(\mathcal{H};n) = t_s(n)$, and*

(2′) *the only graph on n vertices with $t_s(n)$ edges and with no subgraphs in $\mathcal{H}$ is the Turán graph $T_s(n)$.*

The matroid counterpart of this result is valid; the Bose-Burton geometries play the role that the Turán graphs play in Theorem 4.11. This counterpart is given in the following theorem. Here and elsewhere we extend the notation $ex_q(M;n)$ to $ex_q(\mathcal{M};n)$ for a set $\mathcal{M}$ of GF(q)-representable geometries as you would expect:

$$ex_q(\mathcal{M};n) = \max\{|S| \,:\, \mathrm{PG}(n-1,q)|S \text{ has no member of } \mathcal{M} \text{ as a subgeometry}\}.$$

Theorem 4.12. *Let m be a positive integer and let $\mathcal{M}$ be a set of geometries that are all representable over the field* GF(q) *and such that each geometry in $\mathcal{M}$ has critical exponent at least m over* GF(q). *Assume that for some n_0 with $n_0 > m$, the following conditions hold:*

(1) $ex_q(\mathcal{M};n_0) = (q^{n_0} - q^{n_0-m+1})/(q-1)$, *and*
(2) *the only subgeometry of* $\mathrm{PG}(n_0-1,q)$ *with* $(q^{n_0} - q^{n_0-m+1})/(q-1)$ *points and with no subgeometries in $\mathcal{M}$ is* $\mathrm{PG}(n_0-1,q)\backslash\mathrm{PG}(n_0-m,q)$.

Then for all n with $n \geq n_0$, the following statements hold:

(1′) $ex_q(\mathcal{M};n) = (q^{n} - q^{n-m+1})/(q-1)$, *and*
(2′) *the only subgeometry of* $\mathrm{PG}(n-1,q)$ *with* $(q^{n} - q^{n-m+1})/(q-1)$ *points and with no subgeometries in $\mathcal{M}$ is* $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$.

Before we prove this result, we need a technical lemma. Let X be a flat of rank $n-m+1$ in $\mathrm{PG}(n-1,q)$. Each hyperplane of $\mathrm{PG}(n-1,q)\backslash X$ has the form $Y-X$ where Y is a hyperplane of $\mathrm{PG}(n-1,q)$ (with one exception, namely $X=Y$, the converse is true). Now since Y is a hyperplane, the intersection $Y \cap X$ is either X (if $X \subseteq Y$) or a flat of rank $n-m$. It follows that any hyperplane of $\mathrm{PG}(n-1,q)\backslash X$ has one of the following numbers of points:

$$\frac{q^{n-1} - q^{n-m+1}}{q-1} \quad \text{or} \quad \frac{q^{n-1} - q^{n-m}}{q-1}.$$

The second of these is larger. The technical lemma we need asserts that if Z has as many points as X but is not a flat of $\mathrm{PG}(n-1,q)$, then the deletion $\mathrm{PG}(n-1,q)\backslash Z$ has a hyperplane that is larger than any hyperplane of $\mathrm{PG}(n-1,q)\backslash X$. The proof is a simple averaging argument.

Lemma 4.13. *Let X be a set of $(q^{n-m+1}-1)/(q-1)$ elements in* $\mathrm{PG}(n-1,q)$ *that is* not *a flat of* $\mathrm{PG}(n-1,q)$. *Then some hyperplane of the deletion* $\mathrm{PG}(n-1,q)\backslash X$ *has strictly more than $(q^{n-1} - q^{n-m})/(q-1)$ elements. Equivalently, some hyperplane of* $\mathrm{PG}(n-1,q)$ *contains strictly fewer than $(q^{n-m}-1)/(q-1)$ elements of X.*

Proof. Let M be $\mathrm{PG}(n-1,q)\backslash X$. By the Bose-Burton theorem, since X is not a flat of $\mathrm{PG}(n-1,q)$ the geometry M contains at least one flat, say Y, such that the restriction $M|Y$ is isomorphic to $\mathrm{PG}(m-1,q)$. To prove the lemma, it suffices to show that the average number of points of X among hyperplanes of $\mathrm{PG}(n-1,q)$ that contain Y is strictly less than $(q^{n-m}-1)/(q-1)$. This average is computed using the following observations.

(i) The flat Y of $\mathrm{PG}(n-1,q)$ has corank $n-m$ and therefore is contained in $(q^{n-m}-1)/(q-1)$ hyperplanes of $\mathrm{PG}(n-1,q)$.
(ii) There are $(q^{n-m+1}-1)/(q-1)$ points in X and for any x in X the closure $\mathrm{cl}(Y \cup x)$ is a cover of Y.

(iii) Any such closure $\mathrm{cl}(Y \cup x)$ has corank $n-m-1$ in $\mathrm{PG}(n-1,q)$ and therefore is contained in $(q^{n-m-1}-1)/(q-1)$ hyperplanes of $\mathrm{PG}(n-1,q)$.

It follows that the desired average is

$$\frac{\dfrac{q^{n-m+1}-1}{q-1}\,\dfrac{q^{n-m-1}-1}{q-1}}{\dfrac{q^{n-m}-1}{q-1}}.$$

Upon simplification, the desired assertion about the average is the following inequality:

$$\frac{(q^{n-m+1}-1)(q^{n-m-1}-1)}{q^{n-m}-1} < q^{n-m}-1.$$

After some algebraic simplification, this reduces to $0 < (q-1)^2$, which is true, thus proving the lemma. □

A proof of the matroid counterpart of Bollobás' theorem appeared in [1].

Proof of Theorem 4.12. We use n_0 as the base case of an induction. Assume that n exceeds n_0. Recall that the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$ has critical exponent $m-1$ over $\mathrm{GF}(q)$; therefore by Lemma 4.5 no geometry of critical exponent m or greater can be a subgeometry of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$. Since we have assumed that geometries in $\mathcal{M}$ have critical exponent at least m, it follows that $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$ contains no member of $\mathcal{M}$ as a subgeometry. This gives the inequality

$$ex_q(\mathcal{M};n) \geq \frac{q^n - q^{n-m+1}}{q-1}.$$

We claim that it suffices to show the following statement.

(*) The only subgeometry of $\mathrm{PG}(n-1,q)$ with no subgeometries in $\mathcal{M}$ and with $(q^n-q^{n-m+1})/(q-1)$ elements is $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$.

Indeed, after proving this statement, which addresses the case of equality, the proof of the bound follows, as we now show. If there were a subgeometry of $\mathrm{PG}(n-1,q)$ with no subgeometries in $\mathcal{M}$ and with more than $(q^n-q^{n-m+1})/(q-1)$ elements, then there is such a geometry with exactly $\big((q^n-q^{n-m+1})/(q-1)\big)+1$ elements; let $M := \mathrm{PG}(n-1,q)\backslash X$ be such a geometry. Any single-element deletion of M has $(q^n-q^{n-m+1})/(q-1)$ element and no subgeometries in $\mathcal{M}$, so by assertion **(*)**, every single-element deletion of M is isomorphic to $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$; in particular, for every x of $\mathrm{PG}(n-1,q)$ not in X, the set $X \cup x$ is a flat of $\mathrm{PG}(n-1,q)$. This is clearly absurd, so the lemma holds.

To prove assertion **(*)**, assume M has $(q^n-q^{n-m+1})/(q-1)$ elements and no subgeometry in $\mathcal{M}$; let M be $\mathrm{PG}(n-1,q)\backslash X$. If X is not a flat, then by Lemma 4.13, M has a hyperplane H with

$$|H| > \frac{q^{n-1}-q^{n-m}}{q-1}.$$

Thus, $M|H$ is a subgeometry of $\mathrm{PG}(n-2,q)$ that contains no subgeometries in $\mathcal{M}$ and has strictly more than $(q^{n-1}-q^{(n-1)-m+1})/(q-1)$ points, contradicting the induction hypothesis for $n-1$. Thus X must be a flat, as needed to complete the proof. □

Finding a formula for the function $ex_q(M;n)$ for a specific geometry M is, in general, very difficult. In [10], formulas for $ex_q(M;n)$ are found for a limited number of classes of geometries M. The arguments in [10] tend to be fairly intricate and technical, so we will not treat any of these results. All of these results, of course, support Conjecture 4.7. We now pave the way to present Theorem 4.18, which is probably the strongest piece of evidence currently known for Conjecture 4.7. We will use the following elementary lemma.

Lemma 4.14. *If M and M' are both representable over* $\mathrm{GF}(q)$ *and M is a subgeometry of M', then $ex_q(M;n) \leq ex_q(M';n)$.*

Proof. This follows immediately since any geometry that does not contain an M-subgeometry does not contain an M'-subgeometry. □

The following inequality appears in [19].

Lemma 4.15. *Let M be a* $\mathrm{GF}(q)$*-representable geometry with rank m and critical exponent c over* $\mathrm{GF}(q)$*. Then*

$$\frac{q^n - q^{n-c+1}}{q-1} \leq ex_q(M;n) \leq \frac{q^n - q^{n-m+1}}{q-1}.$$

Proof. Since $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-c,q)$ has critical exponent $c-1$ while M has critical exponent c, it follows from Lemma 4.5 that M is not a subgeometry of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-c,q)$. This gives the inequality

$$\frac{q^n - q^{n-c+1}}{q-1} \leq ex_q(M;n).$$

The geometry M, which has rank m and is representable over $\mathrm{GF}(q)$, is a subgeometry of $\mathrm{PG}(m-1,q)$, so by Lemma 4.14 and the Bose-Burton theorem we have

$$ex_q(M;n) \leq ex_q\big(\mathrm{PG}(m-1,q);n\big) = \frac{q^n - q^{n-m+1}}{q-1}$$

thereby completing the proof. □

In particular, Lemma 4.15 says that if the critical exponent c of M exceeds 1, then $ex_q(M;n)$ is bounded below by an exponential function. Nothing is currently known about the case of $c=1$ apart from the results from [10] that we present below. In the case $c=1$, the lower bound in Lemma 4.15 gives no useful information since it is zero. We will show that when q is 2 and M is affine, this lower bound is actually the limit of the ratio $ex_2(M;n)/(2^n-1)$, as would follow from Conjecture 4.7. Before turning to this result, we improve the lower bound in Lemma 4.15 when M is an affine geometry that has the uniform matroid $U_{3,4}$ as a restriction: the next theorem shows that for such M, the function $ex_q(M;n)$ is also bounded below by an exponential function. (As the proof will reveal, we could prove a more general result than this.)

Theorem 4.16. *If M is an affine geometry over* $\mathrm{GF}(q)$ *that has a subgeometry isomorphic to the uniform matroid $U_{3,4}$, then for every integer n with $n \geq 3$, we have the inequality*

$$\frac{6^{1/3} q^{n/3}}{q-1} \leq ex_q(M;n).$$

In particular, for integers n and m with $n, m \geq 3$ we have the inequality

$$\frac{6^{1/3}q^{n/3}}{q-1} \leq ex_q(\mathrm{AG}(m-1,q);n).$$

Theorem 4.16 follows from Lemmas 4.14 and 4.17. The key idea in the proof of Lemma 4.17 is essentially the same as in the proof of the Gilbert-Varshamov bound in coding theory.

Lemma 4.17. *The size function $ex_q(U_{3,4};n)$ is at least $k+1$ if the inequality*

$$k(q-1) + \binom{k}{3}(q-1)^3 < q^n - 1$$

holds. In particular, for $n \geq 3$ we have the inequality

$$\frac{6^{1/3}q^{n/3}}{q-1} \leq ex_q(U_{3,4};n).$$

Proof. To show that $ex_q(U_{3,4};n)$ is at least $k+1$, we construct an $n \times (k+1)$ matrix A over $\mathrm{GF}(q)$ that has no column of zeros (so $M[A]$ has no loops), that has no column that is a scalar multiple of another column (so $M[A]$ has no parallel elements), and that has no set of four columns that form a circuit in $M[A]$ (that is, $M[A]$ has no $U_{3,4}$-subgeometry). Construct the matrix A one column at a time, starting with an arbitrary nonzero column. If we have i columns that satisfy these conditions, then the next column can be any nonzero n-tuple over $\mathrm{GF}(q)$ except the $i(q-1)$ nonzero multiples of columns already in A and the $\binom{i}{3}(q-1)^3$ or fewer columns that form a 4-circuit with three columns already in A. There is such a column if the inequality

$$i(q-1) + \binom{i}{3}(q-1)^3 < q^n - 1$$

holds, which proves the first assertion. The second follows from this by an elementary computation. □

The next theorem is one of the main theorems of this part of the talks.

Theorem 4.18. *Let M be a binary affine geometry. Then*

$$\lim_{n\to\infty} \frac{ex_2(M;n)}{2^n-1} = 0.$$

By Lemma 4.14, to prove Theorem 4.18, we may assume M is $\mathrm{AG}(m-1,2)$. In particular, we need an upper bound for $ex_2(\mathrm{AG}(m-1,2);n)$ that is considerably smaller than 2^n-1. This is provided by Lemma 4.19, which improves the upper bound in Lemma 4.15 when q is 2 and M is affine.

Lemma 4.19. *For $n \geq m \geq 3$, we have $ex_2(\mathrm{AG}(m-1,2);n) < 2^{n\,t_m+1}$ where $t_m = 1 - 1/2^{m-2}$.*

Proof. The proof is by induction on m. Let M be the rank-n binary geometry $\mathrm{PG}(n-1,2)|S$; let s be the number $|S|$ of elements of M. We will say that elements x and y of S *determine* the element z of $\mathrm{PG}(n-1,2)$ if $\{x,y,z\}$ is a line of $\mathrm{PG}(n-1,2)$. Thus, z may or may not be in S.

Note that for M to have no $\mathrm{AG}(2,2)$-subgeometry, that is, no $U_{3,4}$-subgeometry, no two pairs of elements of M can determine the same element of $\mathrm{PG}(n-1,2)$; from this we get the inequality $\binom{s}{2} \leq 2^n - 1$. Since n, the rank of M is at least three, s must also be at least three, so the inequality $s^2/3 \leq \binom{s}{2}$ is valid. From this we get the inequality $s^2/3 < 2^n$, and so $s < 2^{(n+\log_2 3)/2}$. This establishes the case $m = 3$ since $\log_2 3 < 2$.

Now assume that m is at least three and that the lemma holds for m. The key to the induction step for case $m+1$ is to show, as in the base case just treated, that each element z of $\mathrm{PG}(n-1,2)$ must not be determined by too many pairs of elements of M. Toward this end, assume that some element z of $\mathrm{PG}(n-1,2)$ is determined by at least $2^{(n-1)t_m+1}$ pairs of elements of M in the case that m exceeds 3, or by at least $2^{(n-1+\log_2 3)/2}$ pairs of elements of M in the case of $m = 3$. Let Z be the set of elements that occur in pairs that determine z, let M' be the restriction of $\mathrm{PG}(n-1,2)$ to the set $Z \cup z$, and let N be the simplification of the contraction M'/z. Since the lines through z in M' give rise to the elements of N, it follows that N, which has rank $n-1$ or less, has at least $2^{(n-1)t_m+1}$ elements in the case of $m > 3$, or at least $2^{(n-1+\log_2 3)/2}$ elements in the case of $m = 3$. It follows from the induction hypothesis that N has a restriction, say $N|T$, that is isomorphic to the affine geometry $\mathrm{AG}(m-1,2)$. The elements of T arise from pairs of elements of M that determine z; let T' be the set of elements in these pairs. Thus the geometry $M|T'$ has twice as many elements as $N|T$, that is, 2^m elements. Also, $M|T'$ has rank $m+1$. Furthermore, $M|T'$ has no lines with three elements, for a pair of elements in T' either determines z, which is not in T', or the pair corresponds to a pair of elements in $N|T$, which is $\mathrm{AG}(m-1,2)$ and so has only two elements on a line. It follows from the theorem of Bose and Burton that the subgeometry $M|T'$ of M is the affine geometry $\mathrm{AG}(m,2)$. We conclude that in order for M to have no subgeometry isomorphic to $\mathrm{AG}(m,2)$, it follows that each element in $\mathrm{PG}(n-1,2)$ must be determined by fewer than $2^{(n-1)t_m+1}$ pairs of elements in M in the case of $m > 3$, or fewer than $2^{(n-1+\log_2 3)/2}$ pairs of elements of M in the case of $m = 3$. Therefore, if M has no $\mathrm{AG}(m,2)$-subgeometry, we must have the inequality

$$\binom{s}{2} < (2^n - 1)2^{(n-1)t_m+1},$$

while for $m = 3$, we get the stronger inequality

$$\binom{s}{2} < (2^n - 1)2^{(n-1+\log_2 3)/2}.$$

Now replace $\binom{s}{2}$ by $s^2/3$ and manipulate as above. The resulting inequalities are

$$s < 2^{n\,t_{m+1}+(\frac{1}{2})^{m-1}+\frac{1}{2}\log_2 3}$$

and

$$s < 2^{\frac{3}{4}n-\frac{1}{4}+\frac{3}{4}\log_2 3},$$

respectively. These yield the inequality of the lemma since $(\frac{1}{2})^{m-1} + \frac{1}{2}\log_2 3 < 1$ for $m \geq 4$ and $-\frac{1}{4} + \frac{3}{4}\log_2 3 < 1$. □

The binary affine plane $\mathrm{AG}(2,2)$ is the uniform matroid $U_{3,4}$. The corollary below follows from Lemmas 4.17 and 4.19.

Corollary 4.20. *For $n \geq 3$, we have $6^{\frac{1}{3}}2^{\frac{n}{3}} \leq ex_2(\mathrm{AG}(2,2);n) \leq 2^{\frac{n}{2}+1}$.*

Open Problem 4.21. *Find $ex_2(\mathrm{AG}(2,2);n)$ exactly or at least narrow the gap in Corollary 4.20.*

We single out a special case of Conjecture 4.7 that could be a reasonable next step toward resolving the full conjecture. Even solving this problem in the case when q is 3 would be of considerable interest.

Open Problem 4.22. *Find an upper bound for $ex_q(U_{2,q};n)$ that has order of magnitude significantly less than q^{n-1}.*

To give some idea of the likely level of difficulty of Conjecture 4.7, we mention that this conjecture is strongly related to very difficult, classical problems in projective geometry. A *cap* in $\mathrm{PG}(n-1,q)$ is a set of points no three of which are collinear. Geometers denote the maximum cardinality of a cap in $\mathrm{PG}(n-1,q)$ by $m_2(n-1,q)$. In our notation, $m_2(n-1,q)$ is $ex_q(U_{2,3};n)$. Note that $ex_2(U_{2,3};n)$ is 2^{n-1}; this is a special case of the Bose-Burton theorem. For values of q other than 2, the number $ex_q(U_{2,3};n)$ is very difficult to compute. The order of magnitude of the best currently known upper bounds on $ex_q(U_{2,3};n)$ is q^{n-2} while the order of magnitude of the best currently known lower bounds is $q^{2n/3}$. We cite two specific results along these lines. In 1999, Storme, Thas, and Vereecke showed that for any prime power q with $q \geq 37$, the inequality

$$ex_q(U_{2,3};n) < \frac{2641}{2700}q^{n-2} - \frac{79}{2700}q^{n-3} + \frac{346}{135}q^{n-4} + \frac{67}{27}q^{n-5} - 2\frac{q^{n-5}-1}{q-1} + 1$$

holds. In 1959, Segre proved the inequality

$$ex_q(U_{2,3};3k+1) \geq \frac{q^{2k+2}-1}{q^2-1}.$$

The conjectured counterparts of the Erdös-Simonovits theorem and the Erdös-Stone theorem are very tantalizing and almost surely quite difficult to prove. I will mention another problem, also inspired by a result of extremal graph theory, that should be much more accessible. (I have given some limited attention to this problem, and have verified the conjecture in special cases, but currently I do not have a proof of the conjecture.) We start with a natural extension of Turán's theorem which is not nearly as powerful at the Erdös-Stone theorem but which is still very nice. The theorem is about the subgraphs a graph must have if the number of edges exceeds Turán's bound by one.

Theorem 4.23 (Dirac, 1963)**.** *Any graph on n vertices that has $t_{r-1}(n)+1$ edges has a subgraph isomorphic to $K_{r+1}-e$, a single-edge deletion of K_{r+1}.*

This is a slight improvement over what Turán's theorem states since $K_{r+1}-e$ has two K_r-subgraphs.

In light of the Bose-Burton theorem, it would be natural to conjecture that among subgeometries of $\mathrm{PG}(n-1,q)$ that have

$$\frac{q^n - q^{n-m+1}}{q-1} + 1$$

elements (i.e., one more than the bound in the Bose-Burton theorem), the geometry that has the has the fewest number of $\mathrm{PG}(m-1,q)$-subgeometries is that obtained by adding one more element to the Bose-Burton geometries, that is, $\mathrm{PG}(n-1,q)\backslash X$ where $\mathrm{PG}(n-1,q)|X$ is isomorphic to a single-element deletion of $\mathrm{PG}(n-m,q)$. An easy counting argument, similar to that used in Section 2.4 to count the number of k-dimensional subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$, shows that such a deletion $\mathrm{PG}(n-1,q)\backslash X$ has exactly $q^{(n-m)(m-1)}$ restrictions isomorphic to $\mathrm{PG}(m-1,q)$. This motivates the following conjecture.

Conjecture 4.24. *Let M be $\mathrm{PG}(n-1,q)|S$ or, equivalently, $\mathrm{PG}(n-1,q)\backslash T$. Assume*

$$|S| = \frac{q^n - q^{n-m+1}}{q-1} + 1.$$

Then M has at least $q^{(n-m)(m-1)}$ restrictions isomorphic to $\mathrm{PG}(m-1,q)$. Furthermore, M has exactly $q^{(n-m)(m-1)}$ restrictions isomorphic to $\mathrm{PG}(m-1,q)$ if and only if $\mathrm{PG}(n-1,q)|T$ is isomorphic to a single-element deletion of $\mathrm{PG}(n-m,q)$.

If one can solve this conjecture, the natural next step is to add more than 1 to the bound in the Bose-Burton theorem; adding integers that are the sizes of projective geometries should be especially interesting.

A starting point for additional problems in this part of extremal matroid theory is the literature of extremal graph theory; much inspiration can be found, for instance, in [4] and [5]. We should mention another basic result of extremal graph theory since the matroid counterpart is already known. Among graphs on n vertices with no K_r-subgraph, Turán's graph $T_{r-1}(n)$ is known to be not only the unique graph on n vertices with the maximal number of edges (that is, K_2-subgraphs), but also the unique graph with the maximal number of K_3-subgraphs, K_4-subgraphs, and so on up to K_{r-1}-subgraphs. The following counterpart appears in [1].

Theorem 4.25. *Assume $1 \leq j \leq m-1$ and let M be a subgeometry of $\mathrm{PG}(n-1,q)$ that contains no $\mathrm{PG}(m-1,q)$-subgeometry. The greatest number of subgeometries M can have that are isomorphic to $\mathrm{PG}(j-1,q)$ is*

$$q^{(n-m+1)j}\begin{bmatrix} m-1 \\ j \end{bmatrix}_q = q^{(n-m+1)j}\prod_{i=1}^{j}\frac{q^{m-i}-1}{q^i-1}.$$

Furthermore, M has this maximum number of $\mathrm{PG}(j-1,q)$-subgeometries if and only if M is $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$.

Proof. The Bose-Burton theorem is the base case, $j=1$, for induction. Assume the result holds for $j-1$. For i in $\{j-1,j\}$, let $\mathcal{F}_i$ be the set of rank-i flats F of M for which $|F|$ is $(q^i-1)/(q-1)$. Thus, the inductive assumption is the inequality

$$(3) \qquad |\mathcal{F}_{j-1}| \leq q^{(n-m+1)(j-1)}\prod_{i=1}^{j-1}\frac{q^{m-i}-1}{q^i-1}$$

and the assertion that inequality (3) is strict unless M is $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$. Let P be the set of pairs (F,F') with $F \in \mathcal{F}_{j-1}$, $F' \in \mathcal{F}_j$, and $F \subset F'$. The inductive step will follow by comparing an equality and an inequality involving $|P|$.

Since each member of $\mathcal{F}_j$ contains $\left[\begin{smallmatrix} j \\ j-1 \end{smallmatrix}\right]_q$, or $\left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right]_q$, members of $\mathcal{F}_{j-1}$, we have

$$(4) \qquad |P| = |\mathcal{F}_j| \cdot \frac{q^j-1}{q-1}.$$

Fix $F \in \mathcal{F}_{j-1}$. Let X be $X_1 \cup X_2 \cup \cdots \cup X_k$ where $X_1, X_2, \ldots, X_k$ are the flats F' with (F,F') in P. We claim that the following inequality holds:

$$(5) \qquad k \leq q^{n-m+1}\frac{q^{m-j}-1}{q-1}.$$

Note that k is the number of rank-1 flats in the minor $M|X/F$ of M. We have $r(M|X/F) \leq n-j+1$ since $r(X) \leq n$ and $r(F) = j-1$. We claim that $M|X/F$ has no flat Z such that the simplification of $M|X/F|Z$ is $\mathrm{PG}(m-j,q)$; if there were such a Z, then $Z \cup F$ has rank $(m-j+1)+(j-1)$, that is, m, in M, and Z is a union of $(q^{m-j+1}-1)/(q-1)$ covers of F in $\mathcal{F}_j$, each of which has q^{j-1} points not in F, so

$$|Z \cup F| = \frac{q^{m-j+1}-1}{q-1}q^{j-1} + \frac{q^{j-1}-1}{q-1},$$

that is, $|Z \cup F| = (q^m-1)/(q-1)$, thereby contradicting the assumption that M has no $\mathrm{PG}(m-1,q)$-subgeometry. Since $M|X/F$ has k points, rank at most $n-j+1$, and no $\mathrm{PG}(m-j,q)$-subgeometry, from the Bose-Burton theorem we get the inequality $k \leq (q^{n-j+1} - q^{(n-j+1)-(m-j+1)+1})/(q-1)$, which simplifies to inequality (5).

By inequalities (3) and (5), we have the inequality

$$|P| \leq q^{n-m+1}\frac{q^{m-j}-1}{q-1}q^{(n-m+1)(j-1)}\prod_{i=1}^{j-1}\frac{q^{m-i}-1}{q^i-1}.$$

Combining this with equation (4) gives the inequality

$$|\mathcal{F}_j| \cdot \frac{q^j-1}{q-1} \leq \frac{q^{m-j}-1}{q-1}q^{(n-m+1)j}\prod_{i=1}^{j-1}\frac{q^{m-i}-1}{q^i-1},$$

which gives the claimed inequality:

$$|\mathcal{F}_j| \leq q^{(n-m+1)j}\prod_{i=1}^{j}\frac{q^{m-i}-1}{q^i-1}.$$

For equality to hold in this last inequality, we must have equality in inequality (3) (as well as inequality (5)), and by the inductive assumption equality holds in inequality (3) only if M is $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$. This completes the proof. □

5. Excluding the $(q+2)$-Point Line as a Minor

As we saw in the last section, the geometry $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$ has no subgeometry isomorphic to $\mathrm{PG}(m-1,q)$. However, if n exceeds m, then $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$ has many *minors* isomorphic to $\mathrm{PG}(m-1,q)$. Indeed, it follows by counting points that for any element x of $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)$, the simplification of the contraction $\big(\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(n-m,q)\big)/x$ is $\mathrm{PG}(n-2,q)$, which has $\mathrm{PG}(m-1,q)$ as a minor if n exceeds m. Thus, as one would expect, there is a significant difference between excluding a given geometry as a subgeomety and excluding it as a minor. This section of these talks focuses on excluding certain lines as minors; the next section focuses on excluding the Fano plane as a minor.

To see why it is natural to exclude certain lines as minors, we start with some general comments on minor-closed classes of matroids. As we mentioned in Section 2.2, minor-closed classes of matroids can be characterized by excluded minors, that is, the minor-minimal obstructions to being in the class. Characterizations of certain classes of graphs by excluded minors are familiar. Typically the first such result that one learns in graph theory is that a graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor. Another such result is that series-parallel networks are precisely the 2-connected graphs that have no K_4-minor. Probably the most striking result about excluded minors for graphs is the Robertson-Seymour graph-minors theorem: any minor-closed class of graphs can be characterized by a *finite* list of excluded minors. There is no counterpart of the Robertson-Seymour theorem for matroids in general; many important classes of matroids have infinitely many excluded minors. As we will see soon, there is a finite list of excluded minors (indeed, one excluded minor!) for the class of all binary matroids; however, it is currently unknown whether every minor-closed family of binary matroids has a finite collection of excluded minors. Much current research is devoted to such issues.

The first important excluded-minor theorem in matroid theory is W. T. Tutte's characterization of binary matroids. Lines in $\mathrm{PG}(n-1,2)$ have exactly three points, thus the uniform matroid $U_{2,4}$ cannot be a minor of any binary matroid. In [32] Tutte proved that the converse is also true.

Theorem 5.1 (Tutte, 1958)**.** *A matroid is binary if and only if $U_{2,4}$ is not a minor.*

As is fitting for such a fundamental theorem, there are many proofs of Tutte's result. Some of these proofs are aimed solely at this result per se and do not yield further results; others (including Tutte's) are part of a larger theory. In this part of these talks, among the topics we will address is a circle of ideas in extremal matroid theory from which one can deduce Tutte's theorem fairly easily. (There is some technical machinery we will omit that will leave us somewhat short of giving the complete proof of Tutte's theorem, but what we say should convey the spirit of the proof and those who are interested in pursuing this topic will be in a good position to read the remaining details after our discussion. The complete proof from the geometric perspective we will outline can be found in [6].)

We first consider the generalization of the sole excluded minor for binary matroids, $U_{2,4}$. Just as $U_{2,4}$ is an excluded minor for representability over $\mathrm{GF}(2)$, so $U_{2,q+2}$ is an excluded minor for representability over $\mathrm{GF}(q)$; lines in matroids that are representable over $\mathrm{GF}(q)$ can have $q+1$ points but not $q+2$ points. However, only for $q=2$ is $U_{2,q+2}$ the sole excluded minor. The dual of $U_{2,q+2}$, the uniform matroid $U_{q,q+2}$, is another excluded minor for representability over $\mathrm{GF}(q)$ — in the case of $\mathrm{GF}(2)$, the dual is again $U_{2,4}$ — but for q other than 2 there are always excluded minors in addition to $U_{2,q+2}$ and $U_{q,q+2}$. Indeed, in [28] a lower bound of 2^{q-4} is given for the number of excluded minors for representability over $\mathrm{GF}(q)$ and the proof of this result suggests that the actual number of excluded minors is much larger than 2^{q-4}. Excluded-minor characterizations for representability over $\mathrm{GF}(q)$ are known only for $\mathrm{GF}(2)$, $\mathrm{GF}(3)$, and $\mathrm{GF}(4)$; for $\mathrm{GF}(3)$, see [3, 30]; for $\mathrm{GF}(4)$, see [13]. In the early 1970's, Gian-Carlo Rota conjectured that the number of excluded minors for representability over any fixed finite field $\mathrm{GF}(q)$ is finite; considerable progress is being made toward an affirmative resolution of this conjecture based in large part on ideas developed in the Robertson-Seymour graph-minors project.

Our focus in this section will be on a class of matroids that is larger than the class of matroids that are representable over $\mathrm{GF}(q)$; this class is formed by excluding just one of the excluded minors for representability over $\mathrm{GF}(q)$. Specifically, we will be concerned with the minor-closed class of matroids that is formed by excluding $U_{2,q+2}$ as a minor. We introduce some notation to make it easier to talk about such classes. Let $\mathcal{L}(q)$ denote the class of matroids that are representable over $\mathrm{GF}(q)$. Let $\mathcal{U}(q)$ denote the class of matroids that have no $U_{2,q+2}$-minor. For prime powers

q we have $\mathcal{L}(q) \subseteq \mathcal{U}(q)$; equality holds only for $q = 2$. Note however that, unlike $\mathcal{L}(q)$, the class $\mathcal{U}(q)$ is defined and nonempty even when q is not a prime power. For a minor-closed class $\mathcal{C}$ of matroids, let the *size function* $h(\mathcal{C}; n)$ of $\mathcal{C}$ be defined as follows:

$$h(\mathcal{C}; n) := \max\{|S_M| \,:\, M \in \mathcal{C}, r(M) = n, \text{ and } M \text{ is simple}\},$$

where S_M is the ground set of M. Thus, $h(\mathcal{C}; n)$ is the greatest number of points a rank n geometry in $\mathcal{C}$ can have. For example,

$$h\big(\mathcal{L}(q); n\big) = \frac{q^n - 1}{q - 1}.$$

Note that if M is in $\mathcal{L}(q)$ and M' is obtained from M by adding loops and adding elements parallel to the elements of M, then M' is also in $\mathcal{L}(q)$. Thus, there is no upper bound on the number of elements in rank-n matroids in $\mathcal{L}(q)$. The same is true for many other minor-closed classes of matroids such as $\mathcal{U}(q)$, the class of graphic matroids, and the class of cographic matroids. This is why we focus on *simple* matroids in the definition of the size function. Even with this restriction, $h(\mathcal{C}; n)$ need not exist; for instance, if $\mathcal{C}$ is the class of matroids that are representable over the real numbers, then there is no maximum number of points in any rank greater than 1. Thus it would be of interest to know the answer to the following question: *For which classes $\mathcal{C}$ does $h(\mathcal{C}; n)$ exist?* The answer to this question, which is given in Theorem 5.4 below, shows why the classes $\mathcal{U}(q)$ are important. The key to Theorem 5.4 is the following basic result due to Joseph Kung [19].

Theorem 5.2. *Assume that q is an integer exceeding* 1*. A rank-n geometry M in $\mathcal{U}(q)$ has at most*

$$\frac{q^n - 1}{q - 1}$$

points. Furthermore, such a geometry M has $(q^n - 1)/(q - 1)$ points if and only if M is a projective geometry of order q.

We need to treat a few background topics before we can prove the second half of this theorem; before doing that, we focus on the first half and its significance. The proof of the upper bound is an easy inductive argument. The bound trivially holds when n is 0 or 1. Let M be a rank-n geometry in $\mathcal{U}(q)$ on the ground set S and let x be a point of M. Since the simplification of the contraction M/x is a geometry of rank $n-1$ in $\mathcal{U}(q)$, this contraction has at most $(q^{n-1} - 1)/(q - 1)$ points; since these points correspond to the lines of M that contain x, it follows that there are at most $(q^{n-1} - 1)/(q - 1)$ lines of M that contain x. Since each of these lines has at most q points in addition to x, we get

$$|S| \leq 1 + q\frac{q^{n-1} - 1}{q - 1},$$

that is, $|S| \leq (q^n - 1)/(q - 1)$ as claimed.

This proof suggests a simple extension of Theorem 5.2. If we also know that no line has $q + 1$ points — as in the affine geometry $\mathrm{AG}(n - 1, q)$ — then we can strengthen the inequality: there are at most $q - 1$ points on each line in addition to x, so

$$|S| \leq 1 + (q - 1)\frac{q^{n-1} - 1}{q - 1},$$

that is, $|S| \leq q^{n-1}$. This proves the first half of the following theorem from [6].

Theorem 5.3. *Assume that q is an integer exceeding* 1*. A rank-n geometry M in $\mathcal{U}(q)$ that has no lines with $q+1$ points has at most q^{n-1} points. Furthermore, such a geometry M has q^{n-1} points if and only if M is an affine geometry of order q.*

We have proven only half of these theorems. Before proceeding to the parts that require more background, let us mention a result we can prove now along with some very natural conjectures and open problems.

Above we asked: Which minor-closed classes $\mathcal{C}$ of matroids have the size function $h(\mathcal{C};n)$ defined for all n? The following result of Joseph Kung [19] answers this question and points out the important role that the classes $\mathcal{U}(q)$ play in this area of extremal matroid theory.

Theorem 5.4. *For a minor-closed class $\mathcal{C}$ of matroids, the following statements are equivalent.*

(1) *The size function $h(\mathcal{C};n)$ is defined for all n.*
(2) *The size function $h(\mathcal{C};n)$ is defined for $n=2$.*
(3) *The class $\mathcal{C}$ is contained in $\mathcal{U}(q)$ for some integer q exceeding* 1*.*

Proof. That statement (1) implies statement (2) is trivial. That statement (2) implies statement (3) is also trivial; simply let q be $h(\mathcal{C};2)-1$. That statement (3) implies statement (1) follows from Theorem 5.2. □

Part of what is striking about Theorem 5.2 is that although for prime powers q, the geometry $U_{2,q+2}$ is not the only excluded minor for the class $\mathcal{L}(q)$, nonetheless this one excluded minor determines the size function of $\mathcal{L}(q)$. That for prime powers q the maximal number of points for rank-n geometries in the classes $\mathcal{U}(q)$ and $\mathcal{L}(q)$ agree suggests the following conjecture.

Conjecture 5.5. *Let M be a rank-n geometry in $\mathcal{U}(q)$ and let k be an integer with $0<k<n$. The number of flats of rank k in M is at most the Gaussian coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1},$$

that is, the number of rank-k flats in $\mathrm{PG}(n-1,q)$*. Furthermore, the geometry M has $\begin{bmatrix} n \\ k \end{bmatrix}_q$ flats of rank k if and only if M is a projective geometry of order q.*

Although this conjecture is extremely plausible, it appears to be difficult to prove. Theorem 5.2 is the case $k=1$ of this conjecture. It follows from well-known results (e.g., results in [15]) that if the claimed inequality holds for $k=n-1$ and q a prime power, then the assertion about when equality holds would also follow in that case.

Theorem 5.2 suggests that projective geometries play a crucial role in extremal matroid theory. This is what motivates the following tantalizing conjecture from [19].

Conjecture 5.6. *Let q be an integer exceeding* 1 *and let q_* be the greatest prime power not exceeding q. Then for all sufficiently large n we have*

$$h\big(\mathcal{U}(q);n\big) = \frac{q_*^n-1}{q_*-1}.$$

Furthermore, the only rank-n geometries in $\mathcal{U}(q)$ with $(q_*^n-1)/(q_*-1)$ points are projective geometries of order q.

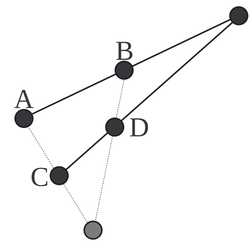

FIGURE 15. The configuration in the Pasch axiom.

This conjecture has been proven only in the case $q = 6$. The following result is from [7].

Theorem 5.7. *Let n be an integer greater than 3 and let M be a rank-n geometry in $\mathcal{U}(6)$ with ground set S. Then*

$$|S| \leq \frac{5^n - 1}{5 - 1}.$$

This upper bound is sharp and is attained only by the rank-n projective geometry $\mathrm{PG}(n-1, 5)$.

Although this result lends support to Conjecture 5.6, the proof of Theorem 5.7 given in [7] relies on some very special results about Latin squares (specifically, that there is no pair of orthogonal Latin squares of order 6), so crucial arguments in that proof do not carry over to the general case.

Shortly we will see several interesting corollaries of Theorems 5.2 and 5.3; first we should complete the proofs of these two theorems. The first step is to give some background in projective and affine geometry. Our treatment of projective and affine geometry in Section 2.4 was based on linear algebra. Geometry, of course, can be approached through axioms. The axiomatic approach to projective geometry starts with the following definition.

Definition 5.8. *A* projective geometry *is a set S of* points *and a collection of subsets of S, the set of* lines*, subject to these axioms:*

(P1) *each pair A, B of distinct points is contained in a unique line, which is denoted $\ell(A, B)$,*
(P2) *if $A, B, C,$ and D are distinct points for which $\ell(A, B) \cap \ell(C, D) \neq \emptyset$, then $\ell(A, C) \cap \ell(B, D) \neq \emptyset$, and*
(P3) *each line contains at least three points.*

Axiom (P2) is the Pasch axiom. It is a way of saying that coplanar lines intersect without mentioning planes; intuitively (and in a sense we could easily make precise), since the intersection $\ell(A, B) \cap \ell(C, D)$ is nonempty, all four points $A, B, C,$ and D lie in a plane, so the lines $\ell(A, C)$ and $\ell(B, D)$ are therefore coplanar and so should intersect nontrivially. This is illustrated in Figure 15.

It is easy to prove from the axioms that all lines in a projective geometry have the same number of points; if this number is $q + 1$, then we call q the *order* of the projective geometry. With this definition, $\mathrm{PG}(n-1, q)$ has order q. (Of course, not all projective geometries have finite order, but those of finite order will be of greatest interest for us.)

The next theorem is one of several results that are commonly called The Fundamental Theorem of Projective Geometry. This result says that, apart from rank 3, the projective geometries we defined algebraically (and their counterparts for skew fields) account for all projective geometries.

Theorem 5.9. *Every projective geometry of rank n, for $n \geq 4$, is isomorphic to* $\mathrm{PG}(n-1, F)$ *for some division ring F.*

The axioms for affine geometry, which we turn to next, are somewhat more complicated than those for projective geometry. (In general, many things are simpler in projective geometry than in affine geometry.)

Definition 5.10. *An* affine geometry *is a set S of* points *and two collections of subsets of S, the set of* lines *and the set of* planes*, subject to these axioms:*

(A1) *each pair A, B of distinct points is contained in a unique line, which is denoted $\ell(A, B)$,*
(A2) *each triple of noncollinear points is contained in a unique plane,*
(A3) *if P is a point not in a line ℓ, then there is a unique line ℓ^* with P in ℓ^* and ℓ parallel to ℓ^* (parallel lines are coplanar and disjoint),*
(A4) *the relation "parallel or equal" is an equivalence relation, and*
(A5) *each line has at least two points.*

Axiom (A3) is the parallel postulate. Note that the reflexive and symmetric properties automatically hold for the relation in axiom (A4); thus, the only issue is the transitive property. Axiom (A5) excludes certain degenerate cases.

It is easy to prove from the axioms that all lines in an affine geometry have the same number of points; if this number is q, then we call q the *order* of the affine geometry. With this definition, $\mathrm{AG}(n-1, q)$ has order q. (Again, we are mainly interested in the case of affine geometries of finite order.)

Affine geometry also has its fundamental theorem, which says that the examples defined algebraically account for all examples, except in rank 3.

Theorem 5.11. *Every affine geometry of rank n, for $n \geq 4$, is isomorphic to* $\mathrm{AG}(n-1, F)$ *for some division ring F.*

Theorems 5.9 and 5.11 can be proven by using geometry to define operations of addition and multiplication and to check that these operations give a division ring that can then be used to coordinatize the geometry. See [2] for a simple presentation of the details.

We are now in a position to complete the proof of Theorem 5.2. Assume that M is a rank-n geometry in $\mathcal{U}(q)$ that has $(q^n-1)/(q-1)$ points. To show that M is a projective geometry of order q we need to show that M satisfies axioms (P1)–(P3). Every matroid satisfies axiom (P1). For M to have $(q^n-1)/(q-1)$ points, all of the inequalities in the proof of the upper bound need to be equalities; specifically, we can conclude that each point of M is in $(q^{n-1}-1)/(q-1)$ lines, each of which has $q+1$ points. Thus, axiom (P3) holds; also, once we have verified (P2) it follows that the order of the projective geometry is q. Axiom (P2) would follow from (and, indeed, is equivalent to) showing that coplanar lines intersect. Thus, assume that ℓ and ℓ' are lines in the plane P. By what we have said, each of ℓ and ℓ' have $q+1$ points. Let x be a point on ℓ not on ℓ' and let $y_1, y_2, \ldots, y_{q+1}$ be the points on ℓ'. Note that there are at least $q+1$ lines through x in P, namely

$$\ell_1 := \mathrm{cl}(\{x, y_1\}), \ell_2 := \mathrm{cl}(\{x, y_2\}), \ldots, \ell_{q+1} := \mathrm{cl}(\{x, y_{q+1}\}).$$

The plane P of M gives rise to the line $P-x$ in the contraction M/x and the $q+1$ lines $\ell_1, \ell_2, \ldots, \ell_{q+1}$ in P through x give rise to $q+1$ points on the line $P-x$ of M/x. Since lines in M/x have at most $q+1$ points, it follows that $\ell_1-x, \ell_2-x, \ldots, \ell_{q+1}-x$ are the only points in the line $P-x$ of M/x, that is, there are no lines in P that contain x other than $\ell_1, \ell_2, \ldots, \ell_{q+1}$. Thus ℓ is one of the lines $\ell_1, \ell_2, \ldots, \ell_{q+1}$, so ℓ indeed intersects ℓ'. Thus, axiom (P2) holds, so M is a projective geometry of order q.

Note that by Theorem 5.9, we can deduce more: if the rank is greater than 3, then M is $\mathrm{PG}(n-1,q)$.

We now complete the proof of Theorem 5.3. Assume M is a rank-n geometry in $\mathcal{U}(q)$ that has q^{n-1} points and at most q points on each line. To show that M is an affine geometry of order q, we need to show that M satisfies axioms (A1)–(A5). Axioms (A1), (A2), and (A5) hold for all matroids. In order for M to have q^{n-1} points, equality must hold in all inequalities that were used to prove the upper bound in Theorem 5.3; it follows that each point of M is in $(q^{n-1}-1)/(q-1)$ lines, each of which has q points. Let H be any hyperplane of M and let x be a point in H. Since the simplification $\mathrm{si}(M/x)$ of the contraction M/x has $(q^{n-1}-1)/(q-1)$ points (that is, the number of lines through x in M), and since this geometry has rank $n-1$ and is in $\mathcal{U}(q)$, by Theorem 5.2 we have that $\mathrm{si}(M/x)$ is a projective geometry of rank $n-1$ and order q. It follows that the hyperplane $H-x$ of M/x has $(q^{n-2}-1)/(q-1)$ flats of rank 1; from this and the fact that each line of M has q points, it follows that H has

$$\frac{q^{n-2}-1}{q-1}(q-1)+1$$

points, that is, q^{n-2} points. By applying this argument repeatedly, it follows that each flat F of M of rank i has q^{i-1} points and that for any x in F the simplification of $M|F/x$ is a projective geometry of rank $i-1$ and order q. In particular, each plane of M has q^2 points. Let $\ell = \{y_1, y_2, \ldots, y_q\}$ be a line of M and let x be a point not on ℓ. The lines $\mathrm{cl}(\{x,y_1\}), \ldots, \mathrm{cl}(\{x,y_q\})$ contain $q(q-1)+1$ points in the plane $\mathrm{cl}(\ell \cup x)$, that is, all but $q-1$ points of this plane; since lines contain q points, the other $q-1$ points in the plane are collinear with x, so there is precisely one line in the plane $\mathrm{cl}(\ell \cup x)$ that contains x and does not meet ℓ, showing that axiom (A3) holds. To treat the only nontrivial case of axiom (A4), assume that ℓ, ℓ', and ℓ'' are distinct lines and that ℓ' and ℓ'' are parallel to ℓ. We first argue that ℓ' and ℓ'' are coplanar. Assume this is not the case. Let N be the rank-4 geometry $M|\mathrm{cl}(\ell \cup \ell' \cup \ell'')$. By what we showed above, N has q^3 points and for any x in N, the geometry $\mathrm{si}(N/x)$ is a projective plane of order q. Let x be a point on ℓ'. Since we have assumed that ℓ' and ℓ'' are not coplanar, the line ℓ^* parallel to ℓ'' through x is distinct from ℓ'. Since the lines of N/x arise from the planes of N through x, the sets $\ell'-x$ and ℓ^*-x are rank-1 flats on the lines $\mathrm{cl}_{N/x}(\ell)$ and $\mathrm{cl}_{N/x}(\ell'')$ respectively. Since $\mathrm{si}(N/x)$ is a projective plane, the lines $\mathrm{cl}_{N/x}(\ell)$ and $\mathrm{cl}_{N/x}(\ell'')$ of N/x must intersect and this point of intersection cannot be either $\mathrm{cl}_{N/x}(\ell')$ or $\mathrm{cl}_{N/x}(\ell^*)$. It follows that there are points a in ℓ and a'' in ℓ'' that are collinear with x. The same argument using a different point y on ℓ' shows that there are points b in ℓ and b'' in ℓ'' that are collinear with y; of course, we cannot have both $a=b$ and $a''=b''$, for otherwise the line $\mathrm{cl}(\{a,a''\})$, which contains the points x and y on ℓ', must be ℓ', contrary to the fact that ℓ' is disjoint from both ℓ and ℓ''. It follows that the (three or four) coplanar points a, b, a'', b'' have at least two points, x and

y, of ℓ' in their closure, which implies that ℓ' is in the plane $\mathrm{cl}(\ell \cup \ell'')$, contrary to the assumption that ℓ' and ℓ'' are not coplanar. Thus, ℓ' and ℓ'' are coplanar. If follows that ℓ' and ℓ'' are parallel, for otherwise, since they are coplanar, they would have a point z in common, and both ℓ' and ℓ'' would be lines through z parallel to ℓ, contrary to the uniqueness criterion we proved as part of verifying axiom (A3). Thus, axiom (A4) holds, completing the proof of Theorem 5.3.

From Theorems 5.2 and 5.3 we will deduce that projective and affine geometries can be characterized by some strikingly simple properties. One key to these characterizations is a very useful result, due to Dennis Higgs, known as the scum theorem.

Theorem 5.12 (The Scum Theorem)**.** *Let N be a minor of a matroid M. Then there is an independent subset X of the ground set of M such that N is a restriction of the contraction M/X and the ranks of N and M/X are the same.*

It follows from the scum theorem that the lattice of flats of N is a sublattice of the upper interval $[\mathrm{cl}(X), S]$ in the lattice of flats of M where the rank of this interval is the rank of N. This is where the name comes from: like scum in a pond, the minors all "float to the top" of the lattice of flats.

The significance of the scum theorem is that it greatly simplifies checking for minors. For instance, in this section we are concerned with matroids with no $U_{2,q+2}$-minors; to verify that a matroid M of rank n has no $U_{2,q+2}$-minor, it suffices to show that there are at most $q+1$ hyperplanes that contain any flat of rank $n-2$. As we will see, this condition is often easy to check.

The proof of the scum theorem is simple and worth presenting. Since deletions and contractions commute, the minor N in the scum theorem can be assumed to be expressed so that the contractions precede the deletions, that is, N has the form $M/Z\backslash Y$. Also, if Z' is a basis of Z, then all elements in $Z - Z'$ are loops in the contraction M/Z', so deleting the element in $Z - Z'$ is equivalent to contracting these elements; that is, $M/Z\backslash Y = M/Z'\backslash\big(Y \cup (Z - Z')\big)$. Thus, we can assume that Z is independent. The key additional assertion in the theorem is that the contractions can be adjusted so that the deletions are rank-preserving. From this perspective we see that in order to prove the scum theorem it suffices to prove the following special case of the theorem which we can then apply to M/Z to get the full result.

Theorem 5.13. *If $M\backslash Y$ has rank $r(M) - t$, then there is an independent subset Y' of Y of rank t such that $M\backslash Y$ is a (rank-preserving) deletion of the contraction M/Y', specifically, $M\backslash Y$ is $M/Y'\backslash(Y - Y')$.*

Proof. Let S be the ground set of M and let B be a basis of $M\backslash Y$. Extend B to a basis $B \cup Y'$ of M. Thus, $|Y'|$ is t and we have the equality

$$r\big((S-Y) \cup Y'\big) = r(S-Y) + t.$$

If follows that all elements of Y' are isthmuses of $M|\big((S-Y) \cup Y'\big)$ and therefore isthmuses of $M|(X \cup Y')$ for all sets X contained in $S - Y$. From this we get the equality

$$r\big(X \cup Y'\big) = r(X) + t$$

for all sets X contained in $S - Y$. Thus, we have $r\big(X \cup Y'\big) - r(Y') = r(X)$ for $X \subseteq S - Y$, that is

$$r_{M/Y'\backslash(Y-Y')}(X) = r_{M\backslash Y}(X)$$

for subsets X of $S - Y$. Therefore we get the equality $M/Y'\backslash(Y - Y') = M\backslash Y$ as claimed since these matroids have the same the rank function. $\square$

The first several characterizations of affine and projective geometries are from [8]. In the proofs, it is convenient to use the terms *copoints* for the hyperplanes of a matroid M and *colines* for the flats of rank $r(M) - 2$.

Theorem 5.14. *Let M be a geometry of rank n on the ground set S where $|S|$ is $(q^n - 1)/(q - 1)$. Assume that all lines of M have at least $q + 1$ points. Then M is a projective geometry of order q.*

Proof. By Theorem 5.2, we need only show that M has no $U_{2,q+2}$-minor. Therefore by the scum theorem it suffices to show that there are $q+1$ copoints over each coline. We first prove that each rank-i flat has exactly $(q^i - 1)/(q - 1)$ points. To prove this assertion, let F and F' be flats with F covering F' and let x be a point in $F - F'$. Thus, we have the equality $F = \mathrm{cl}(F' \cup x)$. By counting the points on lines through x, each of which has at least q points besides x, it follows that there are at least $1 + q|F'|$ points in F. It follows easily that each rank-i flat has at least $(q^i - 1)/(q - 1)$ points. Similarly, if any rank-i flat F had more that $(q^i - 1)/(q - 1)$ points, then the same counting argument applied to a saturated chain of flats from F to S would imply that there are more than $(q^n - 1)/(q - 1)$ points in S contrary to the hypothesis. Thus each rank-i flat has precisely $(q^i - 1)/(q - 1)$ points. Thus there are

$$\frac{q^n - 1}{q - 1} - \frac{q^{n-2} - 1}{q - 1} = \frac{q^n - q^{n-2}}{q - 1}$$

points outside each coline and each copoint over the coline contains

$$\frac{q^{n-1} - 1}{q - 1} - \frac{q^{n-2} - 1}{q - 1} = \frac{q^{n-1} - q^{n-2}}{q - 1}$$

of these points. Thus there are $(q^n - q^{n-2})/(q^{n-1} - q^{n-2})$, or $q + 1$, copoints over each coline, as needed. $\square$

Theorem 5.15. *Assume M is a rank-n geometry with q^{n-1} points in which lines have q points and planes have at least q^2 points. Then M is an affine geometry of order q.*

Theorem 5.15 is the case $j = 2$ of the following theorem.

Theorem 5.16. *Let j be an integer with $2 \leq j \leq n - 2$. Assume M is a rank-n geometry with q^{n-1} points in which lines have at most q points, rank-$(j - 1)$ flats have q^{j-2} points, rank-j flats have q^{j-1} points, and rank-$(j + 1)$ flats have at least q^j points. Then M is an affine geometry of order q.*

Proof. As above, we need only show that there are $q + 1$ copoints over each coline. This follows by the same type of counting as at the end of the proof of Theorem 5.14 once we establish that for each i with $j \leq i \leq n$, each rank-i flat has exactly q^{i-1} points. This assertion is shown by induction on i. The base case is one of the hypotheses of the theorem. Thus, assume that F' is a rank-i flat with at least q^{i-1} points and that F covers F'. Fix a point x in $F - F'$; thus, $F = \mathrm{cl}(F' \cup x)$. Also fix a rank-$(j - 1)$ flat Y in F'. Since $|Y|$ is q^{j-2} and all rank-j flats have q^{j-1} points, it follows that Y is in at least $(q^{i-1} - q^{j-2})/(q^{j-1} - q^{j-2})$ rank-j flats in F'. The point x from $F - F'$ together with any rank-j flat in F' containing Y determines a rank-$(j + 1)$ flat in F. Only points in the rank-j flat $\mathrm{cl}(Y \cup x)$ are in more than one such rank-$(j+1)$ flat and each such rank-$(j+1)$ flat has at least $q^j - q^{j-1}$ points in $F - \mathrm{cl}(Y \cup x)$. Thus F has at least

$$q^{j-1} + \frac{q^{i-1} - q^{j-2}}{q^{j-1} - q^{j-2}}(q^j - q^{j-1}) = q^i$$

points. Thus, for all i with $j \leq i \leq n$ each rank-i flat has at least q^{i-1} points. As in the last proof, from the same type of argument together with the hypothesis that M has q^{n-1} points it follows that for all i with $j \leq i \leq n$ each rank-i flat has exactly q^{i-1} points. As noted at the start, the rest of the argument proceeds as in the proof of Theorem 5.14. □

The following theorem is another result of this type from [8].

Theorem 5.17. *Assume q exceeds 2 and M is a rank-n geometry with q^{n-1} points in which lines have q points and copoints have q^{n-2} points. Then M is an affine geometry of order q.*

Proof. Again we need only show that there are at most $q+1$ copoints over each coline. Assume a coline C is covered by $q+2$ or more copoints. Since there are $(q^{n-1} - |C|)/(q^{n-2} - |C|)$ copoints over C, our assumption gives the inequality $|C| \geq 2q^{n-2}/(q+1)$. Select a point x not in the coline C; say x is in the hyperplane H that contains C. We have assumed that H has q^{n-2} points. There are at least $2q^{n-2}/(q+1)$ lines that contain x and a point of C; each of these lines has $q-1$ points in addition to x and two such lines intersect only in x. It follows that there are at least $1 + (q-1)2q^{n-2}/(q+1)$ points in H. From the assumption that q exceeds 2, we get the inequality

$$1 + (q-1)2q^{n-2}/(q+1) > q^{n-2},$$

which contradicts the assumption that H has q^{n-2} points. This contradiction shows that no coline is covered by $q+2$ or more copoints, as needed. □

The key inequality in the proof of Theorem 5.17 relies on the assumption that q exceeds 2; this does not mean that this assumption is necessary in the statement of the theorem. Thus, we pose the following problem.

Open Problem 5.18. Give a proof or counterexample for the following assertion. *If M is a rank-n geometry with 2^{n-1} points in which lines have two points and copoints have 2^{n-2} points, then M is* $\mathrm{AG}(n-1, 2)$.

The last result of this type that we will present is from [9]. Recall that the cocircuits of a matroid M are the set complements of the hyperplanes of M. To motivate the statement of the next theorem, recall that a cocircuit of $\mathrm{PG}(n-1, q)$ is the ground set of $\mathrm{AG}(n-1, q)$ and so has q^{n-1} elements.

Theorem 5.19. *Let q be an integer exceeding 1. Let M be a rank-n geometry in which each cocircuit has at most q^{n-1} elements and in which all lines have at least $q+1$ points. Then M is a projective geometry of rank n and order q.*

Proof. Since all lines of M have at least $q+1$ points, it follows inductively, as we saw in the proof of Theorem 5.14, that for each i with $1 \leq i \leq n$ each rank-i flat of M has at least $(q^i - 1)/(q-1)$ points. Fix a hyperplane H and a point x not in H. Each of the $|H|$ lines of the form $\mathrm{cl}(\{x, y\})$ with y in H contains $q-1$ points in addition to x in the cocircuit complementary to H; thus, this cocircuit has at least $1 + (q-1)|H|$ points. Since there are at most q^{n-1} points in any cocircuit, it follows that each hyperplane has at most $(q^{n-1}-1)/(q-1)$ points. From these conclusions, it follows that each hyperplane has exactly $(q^{n-1}-1)/(q-1)$ points. From this it follows that for each i with $1 \leq i \leq n$ each rank-i flat of M has exactly $(q^i-1)/(q-1)$ points. From this conclusion, the proof is completed via the same counting argument as used in the proof of Theorem 5.14. □

We now turn to the best known upper bounds on the size function for $\mathcal{U}(q)$ when q is not a prime power. These results are far from the claimed upper bound in Conjecture 5.6. We will prove the following result from [7] which is based on results in [6]. Since this result is of somewhat limited scope, we later mention more general but weaker bounds that one can get with the techniques that we illustrate in the proof of the following result.

Theorem 5.20. *If n is at least three and there is no projective plane of order q, then*

$$h\big(\mathcal{U}(q);n\big) \leq q^{n-1} - q^{n-3} + \frac{q^{n-3}-1}{q-1}.$$

That there is no projective plane of order q can be shown for some q, for instance, by the Bruck-Ryser theorem, which says that if there is a projective plane of order q and if q satisfies the congruence $q \equiv 1, 2 \pmod 4$, then q is a sum of two squares. The Bruck-Ryser theorem implies, for example, that there is no projective plane of order 6, but the result says nothing about either 10 or 12. Using other techniques (in part, tremendous computing power), it has been shown that there is no projective plane of order 10; however, it is still unknown whether there is a projective plane of order 12. The case of 10, which is 1^2+3^2, shows that that converse of Bruck-Ryser theorem is not true. The lack of a simple necessary and sufficient condition for the existence of a projective plane of order q is what limits the scope of Theorem 5.20. (It is a long-standing conjecture that there is a projective plane of order q if and only if q is a power of a prime, but this conjecture remains unproven.)

First note that to prove Theorem 5.20 it suffices to prove the inequality in the rank-3 case for then the general case follows by induction. Indeed, assume that the inequality

$$h\big(\mathcal{U}(q);n-1\big) \leq q^{n-2} - q^{n-4} + \frac{q^{n-4}-1}{q-1}$$

holds. By considering the contraction by a point, it follows that each point x in a rank-n geometry in $\mathcal{U}(q)$ is in at most $q^{n-2} - q^{n-4} + (q^{n-4}-1)/(q-1)$ lines; since each such line has at most q points in addition to x, we get the inequality

$$h\big(\mathcal{U}(q);n\big) \leq 1 + q\left(q^{n-2} - q^{n-4} + \frac{q^{n-4}-1}{q-1}\right),$$

which is exactly the inequality in the statement of the theorem.

Thus, to prove Theorem 5.20 in general it suffices to show that a rank-3 geometry in $\mathcal{U}(q)$ has at most q^2-1 points. We prove this assertion in the following way. We will show that if M is a rank-3 geometry in $\mathcal{U}(q)$ with at least q^2 points, then we can extend M to a rank-3 geometry in $\mathcal{U}(q)$ with q^2+q+1 points. However by Theorem 5.2, a rank-3 geometry in $\mathcal{U}(q)$ with q^2+q+1 points is a projective plane of order q and we have assumed that there is no such projective plane. Thus, from this contradiction it follows that the maximal number of points in any rank-3 geometry in $\mathcal{U}(q)$ is at most q^2-1.

The extension step is shown in two parts. First we show that if M is a rank-3 geometry in $\mathcal{U}(q)$ on the ground set S with $q^2 \leq |S| \leq q^2 + q$, then there is a line, say ℓ, in M with exactly q points. Then we use the line ℓ to extend M to a geometry that is also in $\mathcal{U}(q)$ but that has one more point than M. This extension process is applied enough times so that the result has $q^2 + q + 1$ points. We state these two parts as lemmas.

Lemma 5.21. *Assume that M is a rank-3 geometry in $\mathcal{U}(q)$ on the ground set S with $q^2 \leq |S| \leq q^2 + q$. Then there is a line in M with exactly q points.*

Proof. First note that since each point is in at most $q+1$ lines of M, it follows that any line ℓ with $q+1$ points must intersect every other line ℓ' of M in a point for otherwise a point y in ℓ' would be in at least $q+2$ lines, namely ℓ' along with the lines $\mathrm{cl}(\{y, x\})$ for the $q+1$ points x in ℓ.

If a point x is in a line with exactly q points, there is nothing to show. We claim that any point x that is not in a line with exactly q points is in at least two $(q+1)$-point lines. Indeed, if x were in at most one $(q+1)$-point line, then M would have at most $q+1+q(q-2)$ points, that is, at most q^2-q+1 points, contrary to the assumption that M has at least q^2 points. Thus in the rest of the proof we assume that each point is in at least two $(q+1)$-point lines.

From this assumption, it follows that for each point x there is at least one $(q+1)$-point line that does not contain x. It turn it follows that each point of M is in exactly $q+1$ lines.

We claim that there are q^2+q+1 lines. Consider a point y and two $(q+1)$-point lines ℓ_1 and ℓ_2 that contain y. As just noted, y is on $q+1$ lines. Every line of M that does not contain y contains exactly one of the q points of $\ell_1 - y$ and exactly one of the q points of $\ell_2 - y$. Thus, there are $q+1+q \cdot q$ lines in M.

Since M has fewer than q^2+q+1 points, there is some line ℓ with $q+1-i$ points for some i with $1 \leq i \leq q-1$. Note that each point x that is not in ℓ is in $q+1-i$ lines that intersect ℓ in a point and in i lines that are disjoint from ℓ. Thus there are $\big(|S|-(q+1-i)\big)i$ pairs (x, ℓ') where x is a point not in ℓ and ℓ' is a line that contains x but no point of ℓ. By the assumption on $|S|$, there are at least $\big(q^2-(q+1-i)\big)i$ such pairs. There are $(q+1-i)q$ lines that intersect ℓ in a point, and so $(q^2+q)-(q+1-i)q$, or iq, lines that are disjoint from ℓ. If one of these iq lines is a q-point line, there is nothing to show. If each of these iq lines has at most $q-1$ points, the number of pairs (x, ℓ') as above is at most $iq(q-1)$. Thus

$$\big(q^2-(q+1-i)\big)i \leq iq(q-1).$$

This is equivalent to $i \leq 1$, so $i = 1$. Thus, ℓ is a q-point line, which completes the proof. □

Lemma 5.22. *If M is a rank-3 geometry in $\mathcal{U}(q)$ on the ground set S and if $q^2 \leq |S| \leq q^2+q$, then there is a rank-3 geometry M' in $\mathcal{U}(q)$ on $S \cup e$, with e not in S, such that the deletion $M' \backslash e$ is M.*

Proof. By the last lemma we know that some line of M, say ℓ, has exactly q points. To define M' with ground set $S \cup e$ it suffices to specify the lines of M'; these lines are of the following four types.

(i) The set $\ell \cup e$.
(ii) Lines of M that intersect ℓ in one point.

(iii) Sets of the form $\ell' \cup e$ where ℓ' is a line of M that is disjoint from ℓ.
(iv) Sets of the form $\{e, x\}$ where x is a point not on ℓ that is on exactly q lines of M.

What motivates part (iii) of the definition of the lines of M' is that, as observed at the beginning of the proof of the last lemma, $(q+1)$-point lines, such as $\ell \cup e$, in planes in $\mathcal{U}(q)$ must intersect every other line. Any two points of M' must be on a line, and the role of part (iv) is to account for such lines that do not arise from the first three parts.

We need to show that M' is indeed a matroid (from which it follows immediately that M' is a geometry and that $M' \backslash e$ is M) and that M' is in $\mathcal{U}(q)$. To show that M' is a matroid, one can easily verify the conditions for flats given in Theorem 2.12.

To show that M' is in $\mathcal{U}(q)$, by the scum theorem it suffices to show that each point of M' is in at most $q+1$ lines. This statement holds for any point y of ℓ since the lines of M' that contain y are the lines of M that contain y with the exception that the line ℓ of M is replaced by the line $\ell \cup e$ of M'. For a point y of $S - \ell$, there are two options: if y was on $q+1$ lines of M, then exactly one of these lines does not intersect ℓ and so is augmented by e in M'; if y is on only q lines of M, then those q lines together with the new line $\{y, e\}$ of M' are the only lines in M' that contain y. Now consider the lines of M' that contain e. As part of verifying that M' is a matroid, one shows that the lines of M' that contain e partition the points of S; from items (i)–(iv) above, it follows that two points x and y of S are in the same block of this partition if and only if the line $\mathrm{cl}_M(\{x, y\})$ is ℓ or is disjoint from ℓ. We need to show that there are at most $q+1$ blocks in this partition. First assume that M has a line ℓ' with $q+1$ points. Since every point of $S - \ell'$ is on $q+1$ lines, one for each point of ℓ', it follows that each point of $S - \ell'$ is in the same block as some point of ℓ, so there are at most $q+1$ blocks, as needed. Now assume that M has no lines with $q+1$ points. From Theorem 5.3, it follows that M has exactly q^2 point and is an affine plane of order q. Thus, there are exactly $q-1$ lines that do not meet ℓ (the $q-1$ lines parallel to ℓ) and every point of M is in ℓ or one of the $q-1$ lines parallel to ℓ. It follows that there are exactly q blocks in this case. This completes the proof that M' is in $\mathcal{U}(q)$. □

The same general type of argument as above along with some additional machinery of matroid theory gives the following result from [7] which, although it gives a weaker upper bound than Theorem 5.20, requires no information about the nonexistence of projective planes of a given order. (The additional machinery needed is Henry Crapo's theory of single-element extensions of matroids, of which we saw just a very simple case in the proof above; Chapter 7 of [26] contains an excellent account of the general theory.)

Theorem 5.23. *If n is at least four and q is not a prime power, then*

$$h\big(\mathcal{U}(q); n\big) \leq q^{n-1} - q^{n-4} + \frac{q^{n-4} - 1}{q - 1}.$$

A parity argument in the base case allows us to improve the last two results in the case of odd values of q to obtain the following results from [7].

Theorem 5.24. *If n is at least three and if there is no projective plane of order q where q is odd, then*

$$h\big(\mathcal{U}(q); n\big) \leq q^{n-1} - 2q^{n-3} + \frac{q^{n-3} - 1}{q - 1}.$$

Theorem 5.25. *If n is at least four and q is odd but not a prime power, then*

$$h\big(\mathcal{U}(q);n\big) \leq q^{n-1} - q^{n-3} - 2q^{n-4} + \frac{q^{n-4}-1}{q-1}.$$

The gap between these results and Conjecture 5.6 is huge. It would be of considerable interest to improve these results.

While the last four theorems concern upper bounds on the size function $h\big(\mathcal{U}(q);n\big)$ when q is not a prime power, the same techniques also give representability results. Theorem 5.2 makes it clear that a rank-n geometry with enough points that does not have a $U_{2,q+2}$-minor is automatically representable over the field $\mathrm{GF}(q)$; the next result, from [6], greatly reduces the number of points needed for such a statement to hold.

Theorem 5.26. *Assume that q is a prime power and that n is at least four. Any rank-n geometry in $\mathcal{U}(q)$ with at least q^{n-1} points is representable over $\mathrm{GF}(q)$. If q is odd, then any rank-n geometry in $\mathcal{U}(q)$ with at least*

$$q^{n-1} - \frac{q^{n-2}-1}{q-1}$$

points is representable over $\mathrm{GF}(q)$.

The idea behind the proof of this theorem is to show, as in the proof of Theorem 5.20, that either such a geometry M has the maximal number of points, in which case Theorem 5.2 gives the desired conclusion, or there is a q-point line in M, which can be used to extend M by a single element without introducing a $U_{2,q+2}$-minor. One can iterate the extension step as often as needed to extend M to a geometry without a $U_{2,q+2}$-minor and for which the upper bound of Theorem 5.2 is met, that is, M can be extended to $\mathrm{PG}(n-1,q)$, thereby showing that M is representable over $\mathrm{GF}(q)$. Thus, as in the proof of Theorem 5.20, there are two basic parts, first arguing that there is a q-point line and then extending; the proofs of these steps in general are somewhat more complex than those of Lemmas 5.21 and 5.22 since the results apply in arbitrary ranks, but the basic geometric ideas are essentially the same.

The techniques used to prove Theorem 5.26, which involve only the types of arguments that we have used here together with the theory of single-element extensions, yield a simple geometric proof of Tutte's characterization of binary matroids (Theorem 5.1). In particular, as you can easily check, if M is a rank-n geometry with fewer than 2^n-1 points, then M must have a 2-point line. Thus, the essence of proving Tutte's theorem from this perspective is verifying that an extension result along the lines of Lemma 5.22 applies in this case without any assumption about the number of points. The details can be found in [6].

We close this section with several general observations that are worth keeping in mind as guides to what often happens in extremal matroid theory. We have seen that for prime powers q other than 2, the class $\mathcal{L}(q)$ is properly contained in the class $\mathcal{U}(q)$ yet these two classes have the same size function. Also, although every rank-n geometry in $\mathcal{L}(q)$ can be extended to the unique rank-n geometry in $\mathcal{L}(q)$ that has the maximal number of points, namely, $\mathrm{PG}(n-1,q)$, the same is not true of $\mathcal{U}(q)$; since, for n greater than three, $\mathrm{PG}(n-1,q)$ is the unique rank-n geometry in $\mathcal{U}(q)$ that has the maximal number of points, if every rank-n geometry in $\mathcal{U}(q)$ could be extended to this geometry, then the classes $\mathcal{L}(q)$ and $\mathcal{U}(q)$ would be the same apart from planes, but these is not so. In connection with these ideas we mention the following problem from [19].

Open Problem 5.27. *Classify the extremal geometries in $\mathcal{U}(3)$, that is, the geometries in $\mathcal{U}(3)$ for which no proper extensions of the same rank are in $\mathcal{U}(3)$.*

(See [19] for some of the extremal geometries in $\mathcal{U}(3)$.)

6. Excluding F_7 as a Minor: Heller's Theorem

The Bose-Burton theorem gives the sharp exponential upper bound of $2^n - 2^{n-2}$, or $2^{n-1} + 2^{n-2}$, for the number of points in a rank-n binary geometry with no Fano-subgeometry. Among the results we will see in this part of these talks is that if instead we forbid the Fano matroid F_7 as a *minor* of a binary geometry, then the upper bound on the number of points drops to a quadratic function of n. As in the Bose-Burton theorem, we will be able to identify the geometries that show that the upper bound we obtain is optimal. The basic results in this section go back to I. Heller [16] although we cast them in a very different language (Heller was not writing about matroids) and we will incorporate various refinements and extensions that have been discovered over the years.

We start by showing that binary geometries that do not contain F_7 as a minor must have relatively few 3-point lines through each point.

Theorem 6.1. *Assume that M is a rank-n binary geometry and that there is a point x of M that is in at least n lines that each contain at least three points. Then M has an F_7-minor.*

Proof. Let $\ell_1, \ell_2, \dots, \ell_t$ be the 3-point lines that contain x and let ℓ_i be $\{x, a_i, b_i\}$. We are assuming that the inequality $t \geq n$ holds for which we must also have the inequality $n \geq 3$. Since $\{x, b_1, b_2, \dots, b_t\}$ is a set of more than n points and since M has rank n, it follows that the set $\{x, b_1, b_2, \dots, b_t\}$ is dependent and so contains a circuit, say C. If x is not in C but b_i is in C, then strong circuit elimination implies that there is a circuit C' in $(C \cup \{x, a_i, b_i\}) - b_i$ that contains x. Thus, replacing C by C' and switching the labels of a_i and b_i if needed, we can assume that x is in C; with further relabelling if needed, we can assume that C is $\{x, b_1, b_2, \dots, b_k\}$ where $3 \leq k \leq t$. (Note that k is not 2 since the only 3-circuit that contains $\{x, b_1\}$ is $\{x, a_1, b_1\}$.) We claim that

$$M' := \big(M|(\ell_1 \cup \ell_2 \cup \cdots \cup \ell_k)\big)/\{b_4, b_5, \dots, b_k\}\backslash\{a_4, a_5, \dots, a_k\}$$

is the Fano matroid F_7. The matroid M' has seven elements: $x, a_1, a_2, a_3, b_1, b_2, b_3$, so since M' is binary all we need to show is that M' has rank 3 and seven points (i.e., seven flats of rank 1). From the second part of assertion (iii) of Theorem 2.20, we have that $\{x, b_1, b_2, b_3\}$ is a 4-circuit of M'. From part (v) of Theorem 2.20 and the fact that ℓ_1, ℓ_2, ℓ_3 are lines of M, we get that a_i is in $\mathrm{cl}_{M'}(\{x, b_i\})$ for i in $\{1, 2, 3\}$. Thus, a_1, a_2, a_3 are in $\mathrm{cl}_{M'}(\{x, b_1, b_2, b_3\})$, so $r(M')$ is 3. It follows that M' is F_7 unless any of the following problems arise for some i in the set $\{1, 2, 3\}$: a_i is a loop of M'; a_i and b_i are parallel in M'; a_i and x are parallel in M'. We use part (v) of Theorem 2.20 to show that each of these problems is impossible. If a_i is a loop of M', then a_i is in $\mathrm{cl}_M(\{b_4, b_5, \dots, b_k\})$, which, since $\{x, a_i, b_i\}$ is a circuit of M, implies that x is in $\mathrm{cl}_M(\{b_i, b_4, b_5, \dots, b_k\})$, which is contrary to C being a circuit of M. If a_i and b_i are parallel in M', then a_i is in $\mathrm{cl}_M(\{b_i, b_4, b_5, \dots, b_k\})$, so x is in $\mathrm{cl}_M(\{b_i, b_4, b_5, \dots, b_k\})$, which is contrary to C being a circuit of M.

If a_i and x are parallel in M', then a_i is in $\mathrm{cl}_M(\{x, b_4, b_5, \dots, b_k\})$, so b_i is in $\mathrm{cl}_M(\{x, b_4, b_5, \dots, b_k\})$, which is contrary to C being a circuit of M. Thus, M' is F_7, as needed. □

As we show next, it follows from this theorem that there is a quadratic upper bound on the number of points in rank-n binary geometries that have no F_7-minor.

Corollary 6.2. *A rank-n binary geometry with no Fano-minor has at most* $\binom{n+1}{2}$ *points.*

Proof. The proof is by induction on n. The cases of n being 1, 2, and 3 are immediate since the binomial coefficient $\binom{n+1}{2}$ is 1, 3, and 6, respectively. Assume the bound is true in case $n-1$. Let M be a rank-n binary geometry with no F_7-minor and let x be a point of M. By the induction hypothesis M/x has at most $\binom{n}{2}$ rank-1 flats. Since M has no F_7-minor x is in at most $n-1$ three-point lines; therefore at most $n-1$ rank-1 flats of M/x are pairs of parallel elements. Thus, the number of elements in M is at most $1 + \binom{n}{2} + n - 1$ (accounting for x, the rank-1 flats of M/x, and the rank-1 flats of M/x that could contain a second element); since this sum is $\binom{n+1}{2}$, the induction is complete. □

As we have done with several results in earlier talks, the natural next step (when possible) is to identify the geometries that show that the upper bound is optimal. This classification, together with the results above, is Heller's theorem, recast in matroid terminology; as we will see later in this section, the formulation in Theorem 6.3 is an extension of Heller's original result.

Theorem 6.3. *A rank-n binary geometry with no* F_7*-minor has at most* $\binom{n+1}{2}$ *points; the only such geometry with* $\binom{n+1}{2}$ *points is* $M(K_{n+1})$*, the cycle matroid of the complete graph on* $n+1$ *vertices.*

Proof. By Corollary 6.2 all that remains to show is that any rank-n binary geometry M with $\binom{n+1}{2}$ points and no F_7-minor is $M(K_{n+1})$. By looking at the proof of that corollary we can draw the following conclusion about the geometry M.

(6.3.1) Each point of M is in exactly $n-1$ three-point lines.

The proof of Theorem 6.1 gives the following conclusion.

(6.3.2) The $n-1$ three-point lines through a given point of M span M.

Indeed, if this assertion failed for the 3-point lines $\ell_1, \ell_2, \dots, \ell_{n-1}$ that contain some point x, then we have the inequality $r\big(\mathrm{cl}(\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{n-1})\big) \le n-1$, so Theorem 6.1, applied to $M|\mathrm{cl}(\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{n-1})$, implies that this subgeometry contains an F_7-minor, but we have assumed that M has no such minor.

From statement **(6.3.1)** we can compute the number of 3-point lines of M: multiply the number of points of M by the number of 3-point lines through each point and divide by 3 since each line being counted contains three points. Thus, we have the following expressions for the number of 3-point lines of M:

$$\binom{n+1}{2}\frac{n-1}{3} = \frac{(n+1)n(n-1)}{3!} = \binom{n+1}{3}.$$

Since M has $\binom{n+1}{2}$ points and $n-1$ three-point lines through any point, it follows that for any point x of M the simplification $\mathrm{si}(M/x)$ of the contraction M/x has $\binom{n+1}{2} - 1 - (n-1)$ points, that is, $\binom{n}{2}$ points. Thus, statement **(6.3.1)**, with $n-2$

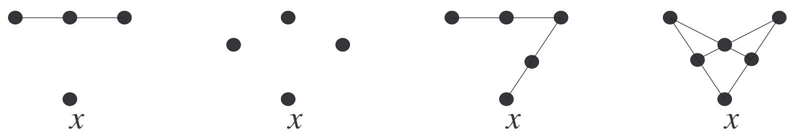

FIGURE 16. The four possible types of planes of M that give rise to three-point lines of $\mathrm{si}(M/x)$.

in place of $n-1$, applies to $\mathrm{si}(M/x)$ so, using the same argument as in the last paragraph, it follows that $\mathrm{si}(M/x)$ has $\binom{n}{3}$ three-point lines.

The $n-1$ three-point lines through x give rise to $n-1$ points in the simplification $\mathrm{si}(M/x)$; in particular they do not give rise to 3-point lines in $\mathrm{si}(M/x)$. The other $\binom{n+1}{3}-(n-1)$ three-point lines through x in M give rise to 3-point lines of $\mathrm{si}(M/x)$ but potentially some pairs of these 3-point lines of M can give the same 3-point line of $\mathrm{si}(M/x)$. Now the 3-point lines of $\mathrm{si}(M/x)$ arise from the planes of M that have three lines that contain x. Since M is binary such planes are restrictions of the Fano plane F_7 and so are of one of the four types shown in Figure 16. Only in the last of the geometries in Figure 16, which is the cycle matroid $M(K_4)$, do two 3-point lines that do not contain x give rise to the same 3-point line of $\mathrm{si}(M/x)$; in this geometry precisely two 3-point lines that do not contain x give rise to one 3-point line of $\mathrm{si}(M/x)$. We will use the phrase *a pair of lines merges* to describe this situation. Thus, the number of 3-points lines of $\mathrm{si}(M/x)$ is the number of 3-point lines of M that do not contain x minus the number of pairs of such lines that merge plus the number of planes of the second type in Figure 16. Note that such merging pairs occur only in planes that contain a pair of 3-point lines through x; since there are $n-1$ three-point lines that contain x, at most $\binom{n-1}{2}$ pairs merge. Subtracting the maximal number of pairs that can merge from the number of 3-point lines that do not contain x gives

$$\left(\binom{n+1}{3}-(n-1)\right)-\binom{n-1}{2}$$

which is

$$\binom{n}{3}+\binom{n}{2}-\binom{n-1}{1}-\binom{n-1}{2}=\binom{n}{3}+\binom{n}{2}-\binom{n}{2}=\binom{n}{3}.$$

Since this is the number of 3-point lines of $\mathrm{si}(M/x)$, it follows that exactly $\binom{n-1}{2}$ pairs of 3-point lines merge. Thus, the $\binom{n-1}{2}$ planes that are spanned by two 3-point lines through x are all isomorphic to $M(K_4)$. (It also follows that there are no planes isomorphic to the second and third planes in Figure 16, but we will not need this fact.)

Let $\ell_1, \ell_2, \dots, \ell_{n-1}$ be the 3-point lines of M that contain x. We have shown that the restriction of M to any flat of the form $\mathrm{cl}(\ell_i \cup \ell_j)$ is $M(K_4)$. Thus, corresponding to each pair ℓ_i, ℓ_j is the unique point, say a_{ij}, in the flat $\mathrm{cl}(\ell_i \cup \ell_j)$ that is not on a 3-point line with x. We claim that, apart from the obvious equality $a_{ij} = a_{ji}$, all such points a_{ij} are distinct. First consider the points a_{ij} and a_{ik}. From statement **(6.3.2)** it follows that the set $\ell_i \cup \ell_j \cup \ell_k$ has rank four. Note that the equalities $\mathrm{cl}(\ell_i \cup \ell_j) = \mathrm{cl}(\ell_i \cup a_{ij})$ and $\mathrm{cl}(\ell_i \cup \ell_k) = \mathrm{cl}(\ell_i \cup a_{ik})$ hold, so if a_{ij} and a_{ik} were the same point, then we would have that $\mathrm{cl}(\ell_i \cup \ell_j \cup \ell_k)$ is $\mathrm{cl}(\ell_i \cup a_{ij})$ from which we get the contradiction that $r(\ell_i \cup \ell_j \cup \ell_k)$ is three. Thus, a_{ij} and a_{ik} must be different. Assume that the equality $a_{hi} = a_{jk}$ holds where $|\{h, i, j, k\}|$ is 4. Since the equalities $\mathrm{cl}(\ell_h \cup \ell_i) = \mathrm{cl}(\ell_h \cup a_{hi})$ and $\mathrm{cl}(\ell_j \cup \ell_k) = \mathrm{cl}(\ell_j \cup a_{jk})$ hold, we also have the equality $\mathrm{cl}(\ell_h \cup \ell_i \cup \ell_j \cup \ell_k) = \mathrm{cl}(\ell_h \cup \ell_j \cup a_{hi})$ from which it follows that $r(\ell_h \cup \ell_i \cup \ell_j \cup \ell_k)$ is four or less; however, from statement **(6.3.2)** it follows that $r(\ell_h \cup \ell_i \cup \ell_j \cup \ell_k)$ is five. Thus, there are $\binom{n-1}{2}$ distinct points a_{ij}. The lines $\ell_1, \ell_2, \ldots, \ell_{n-1}$ and the points a_{ij} account for $1 + 2(n-1) + \binom{n-1}{2}$ points in M; since this sum is $\binom{n+1}{2}$ there are no other points of M.

Let H be the flat $\mathrm{cl}(\ell_1 \cup \ell_2 \cup \cdots \cup \ell_{n-2})$. Since a union of $n-2$ lines through x has rank at most $n-1$, the lines $\ell_1, \ell_2, \ldots, \ell_{n-2}$ cannot span M; however, the lines $\ell_1, \ell_2, \ldots, \ell_{n-1}$ span M so it follows that H is a hyperplane of M. By the same type of counting as at the end of the last paragraph it follows that there are $\binom{n}{2}$ points in H. Therefore there are exactly n points of M, say $p_1, p_2, \ldots, p_n$, not in H. Since each point p_i is on exactly $n-1$ three-point lines, and since each such 3-point line can contain at most one point of H (for otherwise p_i would be in the flat H) it follows that each line $\mathrm{cl}(\{p_i, p_j\})$ is a 3-point line and that the 3-point lines that contain p_i are the lines $\mathrm{cl}(\{p_i, p_j\})$ for j in $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$. By statement **(6.3.2)** it follows that $p_1, p_2, \ldots, p_n$ form a basis of M. Construct a matrix representation A_n for M over GF(2) in which the basis $p_1, p_2, \ldots, p_n$ corresponds to the standard basis and so to an identity submatrix of A_n. Since M is binary, there is no choice about the entries in A_n: with the basis $p_1, p_2, \ldots, p_n$ corresponding to the standard basis vectors, each element y of M other than $p_1, p_2, \ldots, p_n$ is in a unique circuit with a subset of $p_1, p_2, \ldots, p_n$, and in the column representing y there are 1s in the rows that represent the basis elements in this circuit and 0s in the other rows. Since, as shown above, each line $\mathrm{cl}(\{p_i, p_j\})$ is a 3-point line it follows that for each pair i and j with $1 \le i < j \le n$ the matrix A_n has a column with 1s in rows i and j and 0s in the other rows. There are $\binom{n}{2}$ such columns and the hyperplane H has $\binom{n}{2}$ points so these columns, together with the identity submatrix that represents the basis $p_1, p_2, \ldots, p_n$, make up all of A_n. For instance, in the rank-4 case we have the matrix

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

where the first four columns represent p_1, p_2, p_3, p_4. The matrix A_n is precisely the second matrix we obtained in Section 2.3 to show that the cycle matroid $M(K_n)$ is representable over the field GF(2). Thus, since the matrix A_n also represents M over GF(2) it follows that M is isomorphic to $M(K_n)$. □

Theorem 6.3 is certainly a beautiful result of extremal matroid theory but this was not the original context of the result. We should say several things to put this result in a broader context and to suggest some other tools that can be brought to bear in extremal matroid theory.

When we first discussed representability we raised the question of which matroids are representable over all fields. Such matroids are called *regular* matroids. Clearly the class of regular matroids is minor-closed. Several of the most important characterizations of regular matroids are due to W. T. Tutte and are contained in the next two theorems.

Theorem 6.4 (Tutte, 1965)**.** *The following statements are equivalent for a matroid M.*

(1) *M is regular.*
(2) *M can be represented by a totally unimodular matrix, that is, a matrix over $\mathbb{R}$ in which the determinant of every square submatrix is 0, 1, or -1.*
(3) *M is representable over $\mathrm{GF}(2)$ and some field of a characteristic other than 2.*

Totally unimodular matrices play a fundamental role in combinatorial optimization; indeed, there is an important connection between matroid theory and combinatorial optimization that largely revolves around totally unimodular matrices. Regular matroids are sometimes also called unimodular matroids, although in light of statement (2) of the last theorem they should really be called totally unimodular matroids.

Note that regular matroids cannot contain F_7-minors since, as we saw in Theorem 2.29, F_7 is representable only over fields of characteristic 2. Indeed, Tutte proved the following theorem that shows that F_7 plays a very important role for regular matroids.

Theorem 6.5 (Tutte, 1958)**.** *The following statements are equivalent for a matroid M.*

(1) *M can be represented by a totally unimodular matrix.*
(2) *M has no minors isomorphic to $U_{2,4}$, F_7, or F_7^*.*

Thus, regular matroids satisfy the hypothesis of Theorem 6.3. Also, the examples that meet the bound in Theorem 6.3 — cycle matroids of complete graphs — are representable over all fields. From these observations we can deduce the original version of Heller's theorem.

Theorem 6.6 (Heller, 1957)**.** *Let A be an $n \times k$ totally unimodular matrix with no repeated columns. Then $k \leq n^2 + n + 1$. Furthermore, the only such matrix with $k = n^2+n+1$ has as its columns those of the incidence matrix of a complete directed graph on n vertices together with the zero vector, the n standard basis vectors, and their negatives.*

The difference in the upper bounds in Theorems 6.3 and 6.6 is due to allowing the zero column and negatives of columns in Heller's theorem. Note however that what we have shown in Theorem 6.3 is more general than Heller's theorem: the class of binary matroids with no F_7-minor is strictly larger than the class of regular matroids. For instance, F_7^* contains no F_7-minor (proper minors do not have enough points to have an F_7-minor, and F_7 is not self-dual) yet F_7^* is not regular.

It is important to note that Heller's theorem is just the tip of the iceberg; it was discovered about twenty-five years before a much deeper result from which Heller's theorem follows easily. This deeper result, Seymour's decomposition theorem for regular matroids [31], is much too difficult to prove here, but to provide a more balanced view of the field and to suggest some of the deep tools of structural matroid theory that can be brought to bear on questions of extremal matroid theory we should at least state the result, explain the constructions it uses, and show how Heller's result follows easily from this theorem.

Theorem 6.7 (Seymour, 1980)**.** *Every regular matroid M can be constructed from three types of matroids using three operations. The three types of matroids are:*

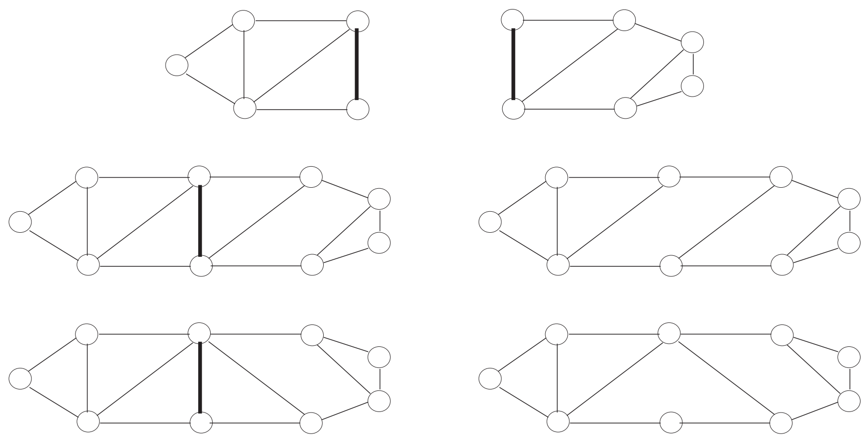

FIGURE 17. Two graphs with an edge in common, their two parallel connections, and the corresponding 2-sums.

graphic matroids, cographic matroids, and the matroid R_{10} *for which a representation over* GF(2) *is given by the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The three operations are: direct sums, 2*-sums, and* 3*-sums.*

Note that the first five columns of the representation of R_{10} form an identity submatrix; the other five columns have a cyclic pattern.

In Section 2.2 we discussed direct sums. We now turn to 2-sums and 3-sum. For 3-sums, we will take advantage of some simplifications that are valid for binary matroids (all of the matroids under discussion are binary); one can define 3-sums more generally but we will avoid the technical details that this involves.

The parallel connection of two graphs glues the graphs together along a common edge; the 2-sum of the graphs is formed by deleting this edge from the parallel connection. Since from the matroid perspective the focus is on the edges (i.e., vertices play no role) there may be two graphs that can be obtained from this construction according to how we identify the vertices of the common edge; however, these two graphs have the same cycle matroid. This construction is illustrated in Figure 17.

Recall that the flats of a graphic matroid $M(G)$ are the edge sets X that have the property that X contains every edge whose vertices are in the same component of the subgraph induced by the edge set X. It follows that X is a flat of the cycle matroid of the parallel connection of the graphs G and G' if and only if the set of edges of X that are in G is a flat of $M(G)$ and the set of edges of X that are in G' is a flat of $M(G')$.

This suggests how to define parallel connection and 2-sums for any pair matroids that have one element in common. Assume that M and M' are matroids with

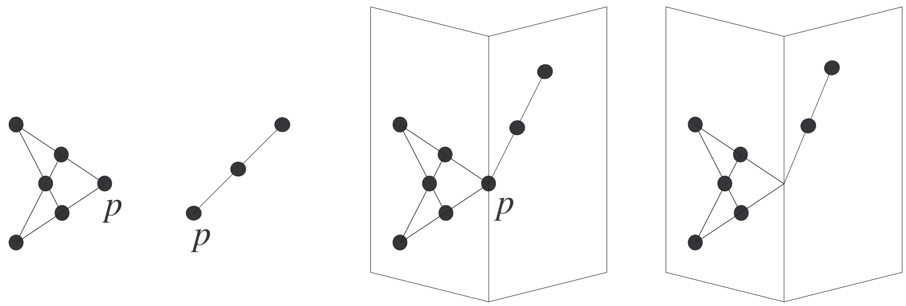

FIGURE 18. Two matroids with an element in common, their parallel connection, and their 2-sum.

ground sets S and S' where $S \cap S' = \{p\}$ and where p is a loop of neither M nor M'. The flats of the *parallel connection* $P(M, M')$ are the subsets X of $S \cup S'$ such that $X \cap S$ is a flat of M and $X \cap S'$ is a flat of M'. The 2-*sum* $M \oplus_2 M'$ of M and M' is $P(M, M')\backslash p$. It is easy to show that these constructions indeed define matroids. These operations are illustrated in Figure 18.

Note that the operation of 2-sum includes the operation of parallel connection as a special case: if we add an element p' parallel to p in M to obtain M'', then $M'' \oplus_2 M'$ is isomorphic to $P(M, M')$.

Some basic facts about 2-sums that follow easily from the definition, and which are relevant to Heller's theorem, are these: the number of elements in $M \oplus_2 M'$ is $|S| + |S'| - 2$ and the rank of $M \oplus_2 M'$ is $r(M) + r(M') - 1$. The first assertion is obvious; the second statement follows easily by considering maximal chains of flats.

The generalized parallel connection of two graphs along a common K_3-subgraph glues the graphs together along the common subgraph; the 3-sum of these graphs is formed by deleting the edges of the K_3-subgraph from this generalized parallel connection. Note that two graphs with a common K_3-subgraph can be glued together in just one way. This construction is illustrated in Figure 19. It is easy to show that a set X of edges is a flat of the cycle matroid of generalized parallel connection of G and G' along a common K_3-subgraph if and only if the set of edges of X that are in G is a flat of $M(G)$ and the set of edges of X that are in G' is a flat of $M(G')$.

Note that the edges of a K_3-subgraph of G give a $U_{2,3}$-subgeometry of the cycle matroid $M(G)$. This suggests how to define the operations of generalized parallel connection and 3-sum for binary matroids that have a common $U_{2,3}$-subgeometry. Assume that M and M' are binary matroids on the ground sets S and S' with $S \cap S' = \{a, b, c\}$ and where the restrictions $M|\{a, b, c\}$ and $M'|\{a, b, c\}$ are both $U_{2,3}$; call this common restriction N. The flats of the *generalized parallel connection* $P_N(M, M')$ are the subsets X of $S \cup S'$ such that $X \cap S$ is a flat of M and $X \cap S'$ is a flat of M'. The 3-*sum* $M \oplus_3 M'$ of M and M' is $P_M(M, M')\backslash\{a, b, c\}$. It is easy to show that these constructions indeed define matroids. These operations are illustrated in Figure 20. (Technical conditions, which we will not mention, are required to extend these constructions beyond the binary case. One can also, with the same technical conditions, define generalized parallel connections and k-sums

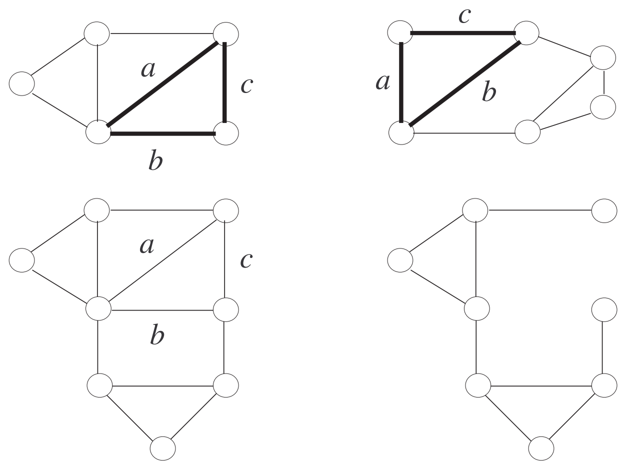

FIGURE 19. Two graphs with a common K_3-subgraph, their generalized parallel connection along the K_3-subgraph, and their 3-sum.

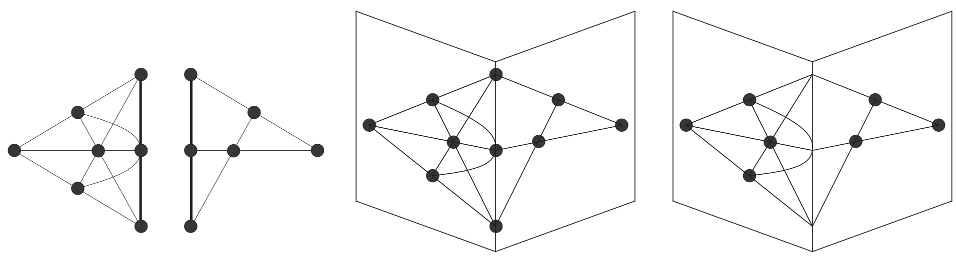

FIGURE 20. Two matroids with a $U_{2,3}$-subgeometry in common, their generalized parallel connection, and their 3-sum.

using common restrictions of higher rank. See Section 4 of Chapter 12 of [26] for more about generalized parallel connections.)

Analogous to what we saw before, the operation of 3-sums includes the operation of generalized parallel connection along a $U_{2,3}$-subgeometry as a special case. Also, we get the following basic facts about 3-sums: for matroids M and M' on the ground sets S and S', the number of elements in $M \oplus_3 M'$ is $|S| + |S'| - 6$ and the rank of $M \oplus_3 M'$ is $r(M) + r(M') - 2$.

With Seymour's decomposition theorem we can sketch a very short proof of Heller's theorem in the case of regular matroids. Graphic geometries of rank n have at most $\binom{n+1}{2}$ points; as we saw in Theorem 3.1, cographic geometries of rank n have at most $3n - 3$ points; the matroid R_{10} has rank 5 and only 10 points. It is easy to show that if we combine two or more such matroids under direct sums, 2-sums, and 3-sums to produce a rank-n regular geometry, the result has *strictly fewer* than $\binom{n+1}{2}$ points. Thus, the maximum number of points in a rank-n regular geometry is $\binom{n+1}{2}$ and this is attained only by the geometry $M(K_{n+1})$.

The argument we just gave is limited to regular matroids; it does not prove the stronger result treated in Theorem 6.3. However, this can be fixed. From Seymour's splitter theorem one can prove the following result. (The splitter theorem is a very powerful structural result that we will not describe; see Chapter 11 of [26] for the statement, proof, and a variety of applications of the splitter theorem.)

Theorem 6.8 (Seymour, 1981)**.** *All binary matroids with no F_7-minor can be constructed from two types of matroids using two operations. The two types of matroids are regular matroids and F_7^*. The two operations are direct sums and 2-sums.*

Since F_7^* has rank 4 but only seven points, from this theorem one can easily prove Theorem 6.3.

One reason we mention results like Theorems 6.7 and 6.8 is to call attention to the fact that a good knowledge of what has been done, or what can be done, in matroid structure theory can contribute greatly toward advances in extremal matroid theory.

To round-out this section we mention several related results and open problems; more theorems and open problems, as well as more details on the ones we mention, can be found in [19] and the references given there.

A number of graph theoretic results can be cast as results in extremal matroid theory. For instance, Euler's upper bound on the number of edges in a simple planar graph on n vertices can be stated as follows: if $\mathcal{P}$ denotes the minor-closed class of cycle matroids of planar graphs, then the size function of $\mathcal{P}$ is the linear function $h(\mathcal{P}; n) = 3n - 3$. Similarly, from the theory of series-parallel networks we have that for the minor-closed class $\mathcal{S}$ of graphic matroids with no $M(K_4)$-minor the size function is the linear function $h(\mathcal{S}; n) = 2n - 1$. Fix an integer m and let $\mathcal{M}$ be the class of graphic matroids with no $M(K_m)$-minor; Mader [23] proved that $h(\mathcal{M}; n)$ is bounded above by a linear function of n, specifically, $h(\mathcal{M}; n)$ is less than $c_m(n + 1)$ where c_m is the constant 2^{m-3}. (The constant in Mader's result was later improved; using arguments from random graph theory, one can show that c_m can be taken to be a small constant times $m\sqrt{\log_2(m)}$.)

Heller's theorem suggests that it should be interesting to study the size function of the minor-closed class of matroids that are representable over GF(q) and that do not have any PG($m-1, q$)-minor for a fixed m. Note that while the size function may be a polynomial as a function of the rank (as in the case of Heller's theorem), it may also be bounded below by an exponential function in the rank. For example, if we consider matroids that are representable over GF(4) and that contain no PG(2, 4)-minor, among such matroids are the binary projective geometries PG($n - 1, 2$); it follows that the size function for this class is bigger than the exponential function 2^n. Determining the size function of such classes in general is probably quite difficult.

Many results concern excluding geometries that are simpler than PG($m - 1, q$). Among such results are the following two theorems due to Joseph Kung [18].

Theorem 6.9. *The size function $h(\mathcal{C}; n)$ for the minor-closed class of matroids that are representable over* GF(q) *and that contain no $M(K_4)$-minor is bounded above by the linear function $(6q^3 - 1)n$.*

Theorem 6.10. *The size function $h(\mathcal{C}; n)$ for the minor-closed class of matroids that are in $\mathcal{U}(q)$ and that contain no $M(K_4)$-minor is bounded above by the linear function $(6q^{q-1} + 8q - 1)n$.*

The size function for ternary matroids with no $M(K_4)$-minor is known exactly; the following theorem is due to James Oxley [25]. (Matroid structure theory provides many of the key tools used in [25].)

Theorem 6.11. *The size function $h(\mathcal{C};n)$ for the minor-closed class $\mathcal{C}$ of ternary matroids that contain no $M(K_4)$-minor is given by*

$$h(\mathcal{C};n) = \begin{cases} 4n-3, & \text{if } n \text{ is odd;} \\ 4n-4, & \text{if } n \text{ is even.} \end{cases}$$

The matroids in $\mathcal{C}$ that show that the bound is optimal are parallel connections of the geometry AG(2,3) *if n is even, and parallel connections of the geometry* AG(2,3) *and a single* 4*-point line if n is odd.*

Most results of the type illustrated in the last several theorems concern excluding certain matroids with a small number of elements; some of these results exclude the cycle matroids of wheels with four or five spokes, the corresponding whirls, or certain Reid geometries. See [19] for a discussion of more of these results. Among the wealth of conjectures in that paper is the following problem.

Conjecture 6.12. *The size function $h(\mathcal{C};n)$ for the minor-closed class of matroids that have none of $U_{2,5}$, $U_{3,5}$, and* PG(4,3) *as a minor is given by $h(\mathcal{C};n) = 2^n - 1$ for sufficiently large n.*

We close this section by touching on Joseph Kung's growth-rate conjecture. (For the precise statement of this conjecture and for its connection with the critical exponent, see [19] and [20].) By Theorem 5.4, the size function of a minor-closed class of matroids is defined for all positive integers n if and only if the class is contained in the class $\mathcal{U}(q)$ for some q. What order of magnitude can the size function of a minor-closed class of matroids in $\mathcal{U}(q)$ have? We have seen that the size function for the classes $\mathcal{L}(q)$ and $\mathcal{U}(q)$, for prime powers q, is given by

$$h\big(\mathcal{L}(q);n\big) = h\big(\mathcal{U}(q);n\big) = \frac{q^n-1}{q-1}.$$

For the class $\mathcal{G}$ of graphic matroids, we have

$$h(\mathcal{G};n) = \binom{n+1}{2};$$

we have seen that the class of regular matroids and the class of binary matroids with no F_7-minor have the same size function. For the class $\mathcal{G}^*$ of cographic matroids, we have $h(\mathcal{G}^*;n) = 3n-3$. As we saw in several of the results cited in the paragraphs above, there are many other minor-closed classes of matroids whose size function is linear in the rank. While the size function in Theorem 6.11 is not linear, its order of magnitude is linear. There are more minor-closed classes of matroids for which the order of magnitude of the size function is quadratic in the rank; the minors of Dowling lattices [12] over a fixed group provide many such classes. Likewise, there are more classes of matroids for which the order of magnitude of the size function is exponential in the rank. However, we currently know of no minor-closed family of matroids for which the order of magnitude of the size function is cubic in the rank, or quartic, or anything other than the three types we have seen. Roughly stated, the growth-rate conjecture says that the order of magnitude of the size function of any minor-closed class of matroids in $\mathcal{U}(q)$ is either exponential, quadratic, or linear in the rank. The following recent result of James Geelen and Geoff Whittle [14], which has a broader scope than Theorems 6.9–6.11, is probably the strongest piece of evidence to date in support of the growth-rate conjecture.

Theorem 6.13. *Let G be a graph and let q be an integer exceeding* 1*. There is a positive integer λ such that any simple matroid M in $\mathcal{U}(q)$ with no $M(G)$-minor has at most $\lambda r(M)$ elements.*

7. Circuits and Cocircuits

Our final topic in extremal matroid theory has several striking differences compared with the results we treated in the earlier talks. First, the main result will apply for all connected matroids rather than for a special minor-closed or subgeometry-closed class of matroids. Second, we have been concerned with bounding the number of *points* (flats of rank 1) *as a function of rank;* our main result will bound the number of *elements without taking the rank into account*. We will be concerned with the existence of certain structures that must be present in a connected matroid whenever the matroid has enough elements, but these structures are not minors. The issues presented here will also lead us back to a theme we saw in Section 5, namely, proving that projective geometries are the unique geometries that show that certain upper bounds in extremal matroid theory are optimal.

We start with a question about graphs. Is it possible for a graph with many edges to have no circuit with more than three edges? Clearly this is possible; among the many examples of such graphs are large forests. To eliminate such examples, it would be natural to consider, for example, only graphs that are 2-connected, that is, connected graphs in which deleting any vertex produces a connected graph, or, equivalently, connected graphs in which each pair of nonloop edges is in a circuit. Again it is easy to think of examples if we go outside the class of simple graphs. For instance, start with a 3-cycle and replace each edge with n edges; the resulting graph, which is not simple, has $3n$ edges and no circuit with more than three edges. Notice that this graph has large minimal edge-cutsets; the graph we just described has minimal edge-cutsets of $2n$ edges. This may make us wonder: Do 2-connected graphs with many edges necessarily have a large circuit or a large minimal edge-cutsets? We will see that this is indeed the case and that the corresponding result holds for matroids.

We now turn to matroids. Recall that the cycle matroid of a graph G is connected if and only if the graph G is 2-connected (in the sense described above) and loopless. In this section we will typically be interested in connected matroids. Recall also that the cocircuits of a matroid M are the circuits of the dual matroid M^*; the cocircuits of M are also the set complements of the hyperplanes of M. As we have seen, in a graphic matroid $M(G)$ the cocircuits correspond to the minimal edge-cutsets of G. Thus, the problem we just posed for graphs can be generalized to connected matroids by considering circuits and cocircuits. This motivates the following problem posed by Robin Thomas during the Graph Minors conference in Seattle in the summer of 1991.

> *Is there an upper bound on the number of elements in a connected matroid in which each circuit has at most c elements and each cocircuit has at most c^* elements?*

In other words, is it true that in any sufficiently large connected matroid, there is either a large circuit or a large cocircuit? Very shortly after the question was posed the upper bound in the following theorem was given by Lovász, Schrijver, and Seymour.

Theorem 7.1. *Let c and c^* be nonnegative integers not both of which are zero. Assume that M is a connected matroid, that every circuit of M has at most c elements, and that every cocircuit of M has at most c^* elements. Then M has at most 2^{c+c^*-1} elements.*

We will soon see that the upper bound in this theorem is far from optimal. However, it is worth seeing the proof of this result. (The optimal result, which we will cite later, is difficult to prove and we will not be able to present that proof here.) We need a few preliminary results, the first of which is a fundamental theorem on connectivity due to W. T. Tutte.

Theorem 7.2. *If x is an element of a connected matroid M, then either the deletion $M\backslash x$ or the contraction M/x is connected.*

Proof. If the deletion $M\backslash x$ is connected, then there is nothing to show, so assume that $M\backslash x$ is disconnected and let y and z be elements in distinct components of $M\backslash x$. By the definition of matroid connectivity the elements y and z are in a circuit C of M. Since y and z are in distinct components of $M\backslash x$, it follows that C must not be a circuit of $M\backslash x$, so x must be in C. It follows from part (iii) if Theorem 2.20 that $C - x$ is a circuit of the contraction M/x, so y and z are in the same component of M/x. From this, it follows easily that M/x has just one component, that is, the contraction M/x is connected. □

It can happen that both $M\backslash x$ and M/x are connected as is true for $U_{3,5}$: deleting any element of $U_{3,5}$ gives $U_{3,4}$ which is connected; contracting any element of $U_{3,5}$ gives $U_{2,4}$ which is also connected. In contrast, in $U_{3,4}$ only the contractions are connected; deleting any single element gives $U_{3,3}$ which is the free matroid on three elements and so is disconnected. In these examples the particular element x did not matter since all elements of a uniform matroid are essentially the same, but this is not true for arbitrary matroids.

Recall from part (iii) of Theorem 2.20 that the circuits of a contraction M/X are the minimal sets of the form $C - X$ as C ranges over the circuits of M. Also, the circuits of a deletion $M\backslash X$ are the circuits of M that are disjoint from X. From these observations together with duality we get the following lemma.

Lemma 7.3. *Each circuit of a minor of M is contained in a circuit of M. Dually, each cocircuit of a minor of M is contained in a cocircuit of M.*

Note that it is not possible for all but one element, say x, of a circuit C of a matroid M to be in a hyperplane of M since x is in the closure of $C - x$ and so in any flat that contains $C - x$. This proves the next lemma, which we will use later in this section.

Lemma 7.4. *Let C be a circuit of a matroid M and let C^* be a cocircuit of M. If $C \cap C^*$ is nonempty, then $|C \cap C^*|$ is at least two.*

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. We induct on the exponent $c + c^* - 1$ in the claimed upper bound. In the base case $c+c^*-1$ is 0, for which there are two options for c and c^*. One option is that c is 0 and c^* is 1; thus, a matroid M satisfying the assumptions of the theorem has no circuits and so contains only isthmuses; since M is connected it follows that M is $U_{1,1}$, which satisfies the stated bound. The other option is that c is 1 and c^* is 0; thus, a matroid M satisfying the assumptions of the theorem has no hyperplanes and so has rank 0; since M is connected it follows that M is $U_{0,1}$, which also satisfies the stated bound.

Now assume that $c + c^* - 1$ is positive and that M satisfies the hypotheses of the theorem. Let S be the ground set of M.

First assume that M satisfies the inequality $r(M) \leq \frac{1}{2}|S|$. Let B be a basis of M and let x be in B. By Tutte's theorem at least one of $M\backslash x$ or M/x is connected. Let y be in $B - x$. Again by Tutte's theorem, in the case that $M\backslash x$ is connected, we have that at least one of $M\backslash x, y$ or $M\backslash x/y$ is connected; in the case that M/x is connected, we have that at least one of $M/x\backslash y$ or $M/x, y$ is connected. In this way it follows that there is a partition of B into two parts X and Y (one of which might be empty) so that the minor $M\backslash X/Y$ is connected. Since B is a basis of M, no hyperplane of M contains B, so each cocircuit of M contains at least one element of B. Since the cocircuits of minors of M are subsets of the cocircuits of M, it follows that each cocircuit of $M\backslash X/Y$ has at most $c^* - 1$ elements. Since each circuit of $M\backslash X/Y$ is contained in a circuit of M, it follows that each circuit of $M\backslash X/Y$ has at most c elements. Thus, by induction we get the inequality

$$|S| \leq 2|S - B| \leq 2(2^{c+(c^*-1)-1}) = 2^{c+c^*-1},$$

as desired.

Now assume that M satisfies the inequality $r(M) > \frac{1}{2}|S|$. Observation 2.8 gives the equality $r(M) + r(M^*) = |S|$ so the inequality $r(M^*) < \frac{1}{2}|S|$ follows. Now M^* is a connected matroid for which every circuit of M^* has at most c^* elements and every cocircuit of M^* has at most c elements. Therefore by what we just showed, applied to M^*, we get the desired inequality $|S| \leq 2^{c+c^*-1}$ thus completing the proof. □

Theorem 7.1 provides an upper bound and so settled the question that Robin Thomas posed. However, the largest connected matroids known in which each circuit has at most c elements and each cocircuit has at most c^* elements have considerably fewer than 2^{c+c^*-1} elements. This led to the question: Might there be an upper bound that is a polynomial in c and c^*? These considerations led to several papers aimed at getting better upper bounds. The first sharp upper bound was obtained in the special case of graphs by Pou-Lin Wu [35] who proved the following result.

Theorem 7.5 (Wu, 1997)**.** *Let G be a loopless 2-connected graph with at least two edges. If a largest circuit of G has c edges and a largest cocircuit of G has c^* edges, then G has at most $cc^*/2$ edges.*

James Reid conjectured that the same upper bound applies for matroids. This conjecture was proven by Manoel Lemos and James Oxley [21]. Examples show that this upper bound is tight. Obviously this result represents an enormous improvement over Theorem 7.1.

Theorem 7.6 (Lemos and Oxley, 2001)**.** *Let M be a connected matroid with at least two elements. If a largest circuit of M has c elements and a largest cocircuit of M has c^* elements, then M has at most $cc^*/2$ elements.*

Note that the statement of Theorem 7.6 can be recast as follows. Let M be a connected matroid on the set S where $|S|$ is at least two. Then we have the inequality $2|S| \leq cc^*$ where c is the largest number of elements in a circuit of M and c^* is the largest number of elements in a cocircuit of M. Thus, the following statement is stronger than Theorem 7.6.

> **(7.6.∗)** *For any 2-connected matroid M on at least two elements in which c is the largest number of elements in a circuit of M, there is a collection $\mathcal{C}^*$ of c cocircuits of M such that each element of M is in at least two cocircuits in $\mathcal{C}^*$.*

Statement (7.6.∗) has been proven for 2-connected graphs by V. Neumann-Lara, E. Rivera-Campo, and J. Urrutia [24]. A proof of statement (7.6.∗) for matroids has recently been announced although no preprint is available at the time that these notes are being prepared.

The proof of Theorem 7.6 is too difficult to present here but to illustrate some of the ideas that are used in this area we will prove the following result of Guoli Ding (mentioned in [27]) that provides an alternative proof of one of the steps in the proof of Theorem 7.6. For an element x of a matroid M, let $c_x(M)$ be the maximum number of elements in a circuit of M that contains x; similarly, let $c_x^*(M)$ be the maximum number of elements in a cocircuit of M that contains x.

Theorem 7.7. *Assume that M is a connected matroid on the ground set S where $|S|$ is at least two. For any element x in S, we have the inequality*

$$|S| \leq (c_x(M) - 1)(c_x^*(M) - 1) + 1.$$

This theorem is a corollary of the following more precise result.

Theorem 7.8. *Assume that M is a connected matroid on the ground set S where $|S|$ is at least two. For any element x in S, there is a collection of at most $c_x(M)-1$ cocircuits of M, each of which contains x, whose union is S.*

Proof. We induct on $c_x(M)$. Since M is connected and contains at least two elements, $c_x(M)$ is at least 2; thus, in the base case $c_x(M)$ is 2 and the assertion says that if M is connected and the only circuits that contain x are 2-circuits, then S itself must be a cocircuit of M, that is, M must have rank 1. This statement is transparent: we are assuming that all elements of M are in 2-circuits with x, that is, parallel to x, so M clearly has rank 1.

Now assume $c_x(M)$ is at least three and let C^* be a cocircuit of M that contains x. Note that M has rank at least two (otherwise $c_x(M)$ would be 2), so C^* is a proper subset of S. Similar to what we saw in the proof of Theorem 7.1, there is partition of $C^* - x$ into two parts X and Y (one of which may be empty) such that the minor $M\backslash X/Y$ of M is connected. Let N denote this minor. Note that N has at least two elements, namely x and an element in the hyperplane complementary to C^*. Recall that each circuit of N is contained in a circuit of M. By Lemma 7.4, circuits and cocircuits cannot intersect in only one element, so any circuit of M that contains x must also contain at least one element in either X or Y. From this, it follows that $c_x(N)$ is at most $c_x(M) - 1$. It follows from the induction hypothesis that there is some collection of cocircuits of N, say $C_1^*, C_2^*, \ldots, C_k^*$ with $k \leq c_x(M)-2$ such that the union $C_1^* \cup C_2^* \cup \cdots \cup C_k^*$ is $S-(C^*-x)$. By Lemma 7.3, each cocircuit C_i^* of N is contained in a cocircuit, say D_i^* of M. It follows that S can be written as the union $C^* \cup D_1^* \cup D_2^* \cup \cdots \cup D_k^*$ of at most $c_x(M)-1$ cocircuits of M, as desired. □

Finally, we combine some of the ideas we have seen above with another theme we have seen, namely, characterizing projective geometries through properties arising in extremal matroid theory. It is striking that, apart from some very small cases, all known examples of matroids that show that the upper bound in Theorem 7.6 is optimal have many parallel elements. This suggests that it might be interesting to consider this general type of problem in the realm of simple matroids. This line of inquiry is further motivated by the following very simple but attractive result from [9].

Theorem 7.9. *Let q be an integer greater than 1. Let M be a rank-3 geometry in which each cocircuit has at most q^2 elements. Then the number of elements in M is at most q^2+q+1. Furthermore, M has q^2+q+1 elements if and only if M is a projective plane of order q.*

Proof. If M had at least q^2+q+2 elements, then, since cocircuits have at most q^2 elements, each line of M would contain at least $q+2$ elements. Consider a line ℓ of M and an element x not in ℓ. The lines $\mathrm{cl}(\{x,y\})$, as y ranges over the elements of ℓ, would contain at least $1+(q+2)q$ elements in the cocircuit complementary to ℓ, contrary to the restriction on the cardinalities of cocircuits. Thus M has at most q^2+q+1 elements.

Assume M has q^2+q+1 elements. Essentially the same argument as above shows that each line of M contains exactly $q+1$ elements and each element of M is in exactly $q+1$ lines. From these observations it follows that each pair of lines of M has nonempty intersection. Thus M satisfies the axioms for projective planes so M is a projective plane of order q. □

This theorem is part of the following more general result from [9] of which we cite just part.

Theorem 7.10. *Let d be a positive integer and assume that M is a rank-3 geometry in which each cocircuit has at most d elements. Then M has at most $d+\lfloor\sqrt{d}\rfloor+1$ elements.*

The complete version of this theorem in [9] classifies all geometries that have the maximal number of elements; most are special deletions of projective geometries.

To get similar results in rank 4 we need to assume connectivity as the next example shows. Assume that q is a prime power. Note that the projective geometry $\mathrm{PG}(3,q)$ of rank 4 has q^3 elements in each cocircuit and q^3+q^2+q+1 elements in all. In contrast, the disconnected rank-4 matroid $U_{2,q^3+1}\oplus U_{2,q^3+1}$ has q^3 elements in each cocircuit and $2q^3+2$ elements in all. Still, with considerably more work than in the rank-3 case, it is possible to prove the following theorems.

Theorem 7.11. *Let $q>1$ be an integer. Let M be a connected rank-4 geometry in which each cocircuit has at most q^3 elements. Then the number of elements in M is at most q^3+q^2+q+1. Furthermore, M has q^3+q^2+q+1 elements if and only if q is a prime power and M is the rank-4 projective geometry $\mathrm{PG}(3,q)$.*

Theorem 7.12. *Let $d\geq 2$ be an integer and assume that d is not a perfect cube. Let M be a connected rank-4 geometry in which each cocircuit has at most d elements. Then M has fewer than $d+d^{2/3}+d^{1/3}+1$ elements.*

There are examples (see [9]) that show that in rank 5 assuming just connectivity is not enough to get a similar result. One needs a stronger condition, namely, that the ground set of the matroid cannot be written as the union of two hyperplanes. With this assumption we get the following result, again with a considerable increase in the level of difficulty of the proof.

Theorem 7.13. *Let $q > 1$ be an integer. Let M be a rank-5 geometry whose ground set is not the union of two hyperplanes and in which each cocircuit has at most q^4 elements. Then the number of points in M is at most $q^4 + q^3 + q^2 + q + 1$. Furthermore, M has $q^4 + q^3 + q^2 + q + 1$ points if and only if q is a prime power and M is the rank-5 projective geometry $\mathrm{PG}(4, q)$.*

We end with an open problem that I find very tantalizing.

Open Problem 7.14. *Find the correct hypothesis so that the last several theorems have counterparts in all ranks.*

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