THE NUMBER OF POINTS IN A COMBINATORIAL GEOMETRY WITH NO 8-POINT-LINE MINORS

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ABSTRACT. We show that when n is greater than 3, the number of points in a combinatorial geometry (or simple matroid) G of rank n containing no minor isomorphic to the 8-point line is at most $\frac{1}{4}(5^n - 1)$. This bound is sharp and is attained if and only if the geometry G is the projective geometry PG$(n-1, 5)$ over the field GF(5).

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1. INTRODUCTION.

Synthetic geometry, the direct study of geometrical configurations without the use of coordinates, bifurcated in the twentieth century into two areas: finite geometry and matroid theory. Finite geometry is the study of projective geometries, in particular, projective planes, and related objects, such as designs. The objects studied in finite geometry have homogeneity or regularity properties which follow or are abstracted from group actions on the objects, so that, roughly speaking, the object "looks the same" from each point. Matroid theory, on the other hand, is concerned with the geometric properties of arbitrary sets of points. In the words of Crapo and Rota, matroid theory "may be considered as a revival of projective geometry in its most synthetic form" [3, Chapter 1].

Matroid theory aims to dispense not only with coordinates but with the ambient space altogether. One way to do this is to replace the properties of the ambient space (which one assumes are preserved under projections) by excluded-minor conditions. For example, no rank-2 contraction of a set of points in the projective space PG$(n-1, q)$ over the finite field of order q can have more than $q+1$ points. Hence, one of the properties of sets of points in PG$(n-1, q)$ is that they have no minors isomorphic to the $(q+2)$-point line $U_{2,q+2}$. This numerical condition, by itself, suffices to determine the expected upper bound on the number of points. More specifically, the following result holds [4, Theorem 4.3].

Theorem 1.1. *Assume G is a rank-n geometry (or simple matroid) containing no minor isomorphic to the $(q+2)$-point line. Then the number $|G|$ of points in G is at most*

$$\frac{q^n - 1}{q - 1} = q^{n-1} + q^{n-2} + \cdots + q + 1.$$

For rank greater than 3, this bound is sharp if and only if q is a prime power; when this is the case, $|G|$ equals the upper bound if and only if G is (isomorphic to) the projective geometry PG$(n-1, q)$.

A natural question arising from this result is: *what is the sharp upper bound when q is not a prime power?* In [1], it is shown that if a rank-n geometry G has no minor isomorphic to the $(q+2)$-point line and there is no projective geometry of rank n and order q, then $|G|$ is at most $q^{n-1} - 1$; if in addition q is odd, then $|G|$ is at most

$$q^{n-1} - \frac{q^{n-2} - 1}{q - 1} - 1.$$

These upper bounds, however, are not sharp and can be improved using an induction argument similar to that in the proof of Lemma 5.4. Starting the induction at $n = 3$, it can be shown that when there is no projective plane of order q,

$$q^{n-1} - q^{n-3} + \frac{q^{n-3} - 1}{q - 1}$$

is an upper bound, as is

$$q^{n-1} - 2q^{n-3} + \frac{q^{n-3} - 1}{q - 1}$$

when q is odd. Without using any information about the non-existence of projective planes of order q, we get the following weaker upper bounds by starting the induction at $n = 4$:

$$q^{n-1} - q^{n-4} + \frac{q^{n-4} - 1}{q - 1}$$

for q in general, and

$$q^{n-1} - q^{n-3} - 2q^{n-4} + \frac{q^{n-4} - 1}{q - 1}$$

for q odd. These tighter upper bounds are most probably not sharp.

In this paper, we derive the sharp upper bound when q equals 6, the first positive integer which is not a prime power.

The geometries having no minor isomorphic to the $(q+2)$-point line form a minor-closed class $\mathcal{U}(q)$. For every prime power q' not exceeding q, all geometries representable over the finite field GF(q') are in $\mathcal{U}(q)$. (For $q > 2$, the class $\mathcal{U}(q)$ also contains many non-representable geometries.) In particular, the rank-n projective geometry PG$(n-1, 5)$ is in $\mathcal{U}(6)$. Hence, there is a rank-n geometry in $\mathcal{U}(6)$ having $\frac{1}{4}(5^n - 1)$ points. The main theorem in this paper asserts that when the rank n is greater than 3, the projective geometry PG$(n-1, 5)$ is the rank-n geometry having the maximum number of points in $\mathcal{U}(6)$.

Theorem 1.2. *Let n be greater than 3 and let G be a rank-n geometry in $\mathcal{U}(6)$. Then*

$$|G| \leq \frac{5^n - 1}{5 - 1} = \tfrac{1}{4}(5^n - 1).$$

This upper bound is sharp and is attained only by the rank-n projective geometry PG$(n-1, 5)$ over the finite field GF(5).

Theorem 1.2 verifies the case $q = 6$ of a conjecture made in [4, p. 35].

The proof of Theorem 1.2 is given in Section 5. Sections 2, 3, and 4 treat several lemmas used in the proof. The final section, Section 6, outlines an alternative proof of the bound in Theorem 1.2.

Although we assume some familiarity with basic matroid theory (see, for example, the classic text [3]), this paper is written so as to be reasonably accessible to any synthetic geometer. If G is a geometry on the point set S, we refer to the flats of ranks $r(S)-1$, $r(S)-2$, and $r(S)-3$ respectively as copoints, colines, and coplanes. The following elementary lemma [4, p. 42] will be used freely.

Lemma 1.3. *Let G be a geometry in $\mathcal{U}(q)$. Then a flat of G having rank k and $(q^k-1)/(q-1)$ points is modular. In particular, 6-point lines are modular in geometries in $\mathcal{U}(5)$ and 7-point lines are modular in geometries in $\mathcal{U}(6)$.*

2. Extensions of geometries.

A lemma from [1] that is fundamental to this paper asserts that under certain conditions, a geometry in $\mathcal{U}(q)$ can be extended to a bigger geometry in $\mathcal{U}(q)$ having the same rank. Before stating this lemma, we recall some notions from Crapo's theory of single-element extensions (see [2] and [3, Chapter 10]). A *single-element extension* $G^+(S \cup \{e\})$ of a geometry $G(S)$ is a matroid such that the restriction $G^+|S$ equals G. Let $\mathcal{M}$ be the collection of flats or closed sets A in the lattice $L(G)$ of flats of G such that the closure of A in G^+ is $A \cup \{e\}$. The set $\mathcal{M}$ is a filter in the lattice $L(G)$. In addition, it satisfies the following property (which is equivalent to the property that $\mathcal{M}$ is closed under intersections of modular pairs of flats): if A and B are flats in $\mathcal{M}$ and $r(A \cap B) = r(A) - 1 = r(B) - 1$, then their intersection $A \cap B$ is also a flat in $\mathcal{M}$. A filter $\mathcal{M}$ in $L(G)$ satisfying this property is called a *modular filter*. Crapo proved that single-element extensions of G are in one-to-one correspondence with the modular filters in $L(G)$.

If the extension G^+ is a geometry, then $\mathcal{M}$ does not contain any point of G. Therefore, if two lines in G intersect in a point, then at most one of the lines is in $\mathcal{M}$. When both G and G^+ are rank-3 geometries, this implies that if the filter $\mathcal{M}$ contains a modular line, then it contains no other line. In the rank-3 case, we also have the following useful counting result: the number of lines containing the point e in the extension G^+ is the sum of the number of lines in $\mathcal{M}$ and the number $|S - \bigcup_{\ell \in \mathcal{M}} \ell|$ of points in G not contained in any line in $\mathcal{M}$.

The next lemma is essentially Lemma 6 in [1].

Lemma 2.1. *Let $G(S)$ be a geometry in $\mathcal{U}(q)$ and let ℓ be a q-point line in G. Suppose that for each coplane X with $X \cap \ell = \emptyset$ and $r(X \cup \ell) = r(X) + 2$, at least one of the following conditions holds:*

(1) *there exists a copoint Y containing X such that the rank-2 interval $[X, Y]$ contains $q + 1$ colines,*
(2) *there exist at least $q^2 - 1$ colines in the rank-3 upper interval $[X, S]$, or*
(3) *every coline in the upper interval $[X, S]$ is contained in $q + 1$ copoints and there are at most $q^2 + q + 1$ copoints in $[X, S]$.*

Then G has an extension $G^+(S \cup \{e\})$ of the same rank in $\mathcal{U}(q)$ in which the q-point line ℓ is extended to the $(q + 1)$-point line $\ell \cup \{e\}$.

The proof of Lemma 6 in [1] can be used to prove this lemma without any changes. The basic idea in the proof is to observe that the union $\mathcal{M}$ of the two sets

$$\{A \in L(G) : A \cap \ell = \emptyset \text{ and } r(A \cup \ell) = r(A) + 1\}$$

(the set of flats which "should" intersect ℓ at a point) and

$$\{A \in L(G) : \ell \subseteq A\},$$

is a modular filter in $L(G)$. Let G^+ be the single-element extension determined by $\mathcal{M}$. To show that G^+ is in $\mathcal{U}(q)$, it suffices, by the scum theorem (see, for example, [3, Chapter 9]), to check that every rank-2 upper interval in G^+ contains at most $q + 1$ points. This is guaranteed by any one of the three conditions in the lemma.

We shall use the cases q equals 5 or 6 of Lemma 2.1. Note that when q equals 6,

$$q + 1 = 7, \quad q^2 - 1 = 35, \quad \text{and} \quad q^2 + q + 1 = 43.$$

Two immediate consequences of Lemma 2.1 are the following.

Corollary 2.2. *Let $G(S)$ be a rank-3 geometry in $\mathcal{U}(6)$ with a 6-point line ℓ and a 7-point line. Then $G(S)$ has a single-element extension to a rank-3 geometry $G^+(S \cup \{e\})$ in $\mathcal{U}(6)$ in which ℓ has been extended to the 7-point line $\ell \cup \{e\}$.*

Corollary 2.3. *Let $G(S)$ be a rank-4 geometry in $\mathcal{U}(6)$ with a 6-point line ℓ. Assume that for each point x of G, there is at least one 7-point line of G not containing x. Then $G(S)$ has a single-element extension to a rank-4 geometry $G^+(S \cup \{e\})$ in $\mathcal{U}(6)$ in which ℓ has been extended to the 7-point line $\ell \cup \{e\}$.*

The next lemma is a special case of Corollary 1 in [1]. It can be proved by an easy *ad hoc* counting argument using Lemma 2.1.

Lemma 2.4. *Let G be a rank-3 geometry in $\mathcal{U}(5)$ with at least 28 points. Then G can be extended to the projective plane PG$(2,5)$.*

We are now ready to prove the following lemma.

Lemma 2.5. *Let $G(S)$ be a rank-3 geometry in $\mathcal{U}(5)$ with at least 28 points and let $G^+(S \cup E)$ be an extension of G to a geometry with 32 points. Then G^+ is not in $\mathcal{U}(6)$.*

Proof. Suppose G has m points, where $28 \leq m \leq 31$. By Lemma 2.4, we may assume that G is PG$(2,5) - X$ for some set X of $31 - m$ points in PG$(2,5)$. Let e be a point in E and let G' be the restriction $G^+|S \cup \{e\}$ of G^+. It suffices to show that either the extension G' of G is a subgeometry of PG$(2,5)$ or it is not in $\mathcal{U}(6)$. If the modular filter $\mathcal{M}$ for the extension G' consists of S alone, then e is on m lines in G', and so G' is not in $\mathcal{U}(6)$. If $\mathcal{M}$ contains a 6-point line, then, since 6-point lines in geometries in $\mathcal{U}(5)$ are modular and G' is a geometry, no other line is in $\mathcal{M}$. Therefore, in this case, e is on $1 + (m-6)$ lines in G', and so again G' is not in $\mathcal{U}(6)$. Thus we may assume that all lines in $\mathcal{M}$ have five or fewer points (and hence, m is strictly less than 31). Let i be the number of lines in $\mathcal{M}$. Since the i lines in $\mathcal{M}$ contain at most $5i$ points in G, we have that e is on at least $i + (m - 5i)$ lines in G'. If $i \leq 5$, it follows that G' is not in $\mathcal{U}(6)$. Therefore we may assume that $i \geq 6$. Since no two lines in $\mathcal{M}$ can have a point in common, there is some point a in X such that the lines in $\mathcal{M}$ are precisely the lines ℓ for which $\ell \cup \{a\}$ is a line of PG$(2,5) - (X - \{a\})$. Thus, up to relabeling, the point e is a. It follows that G' is a subgeometry of PG$(2,5)$. □

Corollary 2.6. *Let G be a rank-4 geometry in $\mathcal{U}(6)$. Let P be a plane in G that has at least 28 points and is in $\mathcal{U}(5)$. Then any point x in the complement $G - P$ is on at most 31 lines in G.*

3. Some preliminary bounds.

Although we prove Theorem 1.2 when the rank is greater than 3, the proof relies on the following upper bound for rank-3 geometries in $\mathcal{U}(6)$.

Lemma 3.1. *A rank-3 geometry in $\mathcal{U}(6)$ has at most 35 points.*

This lemma is a special case of Corollary 2 in [1]. The upper bound given in Lemma 3.1 is probably not sharp. It would be interesting to determine the sharp bound.

Since seven copunctual 6-point lines contain 36 points, Lemma 3.1 implies that there is no such configuration in a rank-3 geometry in $\mathcal{U}(6)$. This observation yields the following result about rank-4 geometries in $\mathcal{U}(6)$.

Corollary 3.2. *Let G be a rank-4 geometry in $\mathcal{U}(6)$, let ℓ be a line of G, and let a be a point on ℓ. Then among the seven or fewer planes in G containing ℓ, there are at most six in which there is a 6- or 7-point line not equal to ℓ through a.*

Proof. Assume the conclusion is false. Let b be a point not equal to a in ℓ. Then the rank-3 geometry G/b formed by contracting b has seven 6- or 7-point lines through x, contradicting our earlier observation. □

The proof of Theorem 1.2 also uses a better upper bound on the number of points in planes in $\mathcal{U}(6)$ that have no 7-point lines. (Strictly speaking, we do not need the full strength of this lemma to prove Theorem 1.2.)

Lemma 3.3. *Let G be a rank-3 geometry in $\mathcal{U}(6)$. If G has no 7-point lines, then $|G| \leq 33$.*

Proof. By Lemma 3.1, $|G|$ is at most 35. Assume $|G| = 35$. Since G has no 7-point lines, we can have 35 points only by having each point incident with six 6-point lines and a single 5-point line. (Note that 35 is $5 \cdot 6 + 4 + 1$; this reflects the six 6-point lines through a given point x, each contributing five points in addition to x, the four points besides x on the 5-point line, and x itself.) This and Lemma 2.1 (using condition 2) yield a geometry in $\mathcal{U}(6)$ with 36 points, contrary to Lemma 3.1.

Now assume $|G| = 34$. Note that each point is on seven lines and there are two types of points in G: type 1 points are incident on six 6-point lines and a single 4-point line; type 2 points are incident on five 6-point lines and two 5-point lines. Let there be n_1 points of type 1 and n_2 points of type 2. To count the number of lines, we start by counting the number of 6-point lines, and to do this, we count the number of pairs consisting of a 6-point line and an incident point. By considering the number of incident 6-point lines at points of the two types, we see that the number of such pairs is $6n_1 + 5n_2$. Thus the number of 6-point lines is $(6n_1 + 5n_2)/6$. Applying these ideas to 5- and 4-point lines also, we see that the number of lines is

$$\frac{6n_1 + 5n_2}{6} + \frac{2n_2}{5} + \frac{n_1}{4} = \frac{5}{4}n_1 + \frac{37}{30}n_2.$$

This is at most $5(n_1+n_2)/4$, or $5 \cdot 34/4$, which is less than 43. Since the number of lines is less than 43, we can apply Lemma 2.1 (using condition 3) to any 6-point line in G, yielding a geometry in $\mathcal{U}(6)$ with 35 points. Since this contradicts what we established in the last case, the lemma has been proven. □

4. ORTHOGONAL LATIN SQUARES.

One reason Theorem 1.2 has a relatively easy proof is that one can apply the following classical result arising from Euler's problem of 36 officers: *there exists no pair of orthogonal Latin squares of order* 6. This non-existence theorem was proved by Tarry [6] in 1900. For a modern proof, see, for example, [5].

A *Latin square* L of order q is a $q \times q$ array $(L_{ij})_{1 \leq i,j \leq q}$ filled with symbols from the set $\{1, 2, \ldots, q\}$ such that each row and each column contains every symbol exactly once. Two Latin squares L and M are *orthogonal* if for each ordered pair (h, k), there is a unique position (i, j) such that $L_{ij} = h$ and $M_{ij} = k$. There are several ways to associate Latin squares with configurations of lines in geometries. We shall use the following construction. Let ℓ_r, ℓ_s, ℓ_t be three $(q+1)$-point lines meeting at the common point a in a rank-3 geometry in $\mathcal{U}(q)$. Let the points other than a on ℓ_r be denoted $r_1, r_2, \ldots, r_q$, and similarly for the other two lines ℓ_s and ℓ_t. Construct a $q \times q$ array L_t by the rule: the entry in row i and column j is k if r_i, s_j, and t_k are collinear. Since any two points in a geometry are on a unique line and $(q + 1)$-point lines are modular, L_t is a Latin square.

Lemma 4.1. *Let a be a point of a rank-3 geometry G in $\mathcal{U}(6)$. If a is on a 7-point line, then there are at most two other lines in G through a with more than 5 points.*

Proof. By Corollary 2.2, every 6-point line through a can be extended to a 7-point line through a in some extension of G. Therefore it suffices to show that there can be at most three 7-point lines through a. However, if $\ell_r, \ell_s, \ell_t, \ell_u$ are four 7-point lines through a, then, because a point t_h in ℓ_t and a point u_k in ℓ_u determine a unique line and this line intersects the lines ℓ_r and ℓ_s at points r_i and s_j, the 6×6 Latin squares L_t and L_u are orthogonal, contradicting Tarry's theorem. □

Corollary 4.2. *Let a be a point of a rank-4 geometry G in $\mathcal{U}(6)$. There are at most seven 7-point lines through a.*

Proof. Let ℓ be a 7-point line through a. By contracting any point in $\ell - a$ and applying Lemma 4.1, it follows that at most three of the planes through ℓ contain 7-point lines through a. By Lemma 4.1, each of these planes contains at most two 7-point lines through a in addition to ℓ. □

5. The proof of the main theorem.

We now give the proof of Theorem 1.2, which is by induction. To establish the case $n = 4$, we use three cases: G has a 7-point line; G has no 7-point line but G has a point on seven coplanar lines; G has no point on seven coplanar lines.

Lemma 5.1. *Let G be a rank-4 geometry in $\mathcal{U}(6)$ and assume that the point a of G is on at least one 7-point line ℓ in G. Then $|G| \leq 155$.*

Proof. By Corollary 3.2, at most six of the planes through ℓ can contain 6- or 7-point lines through a. Since ℓ is a 7-point line through a, Lemma 4.1 implies that each plane through ℓ contains at most two other 6- or 7-point lines through a.

First assume that ℓ is the only 7-point line through a. There are at most twelve 6-point lines through a, namely, at most two in each of six or fewer planes through ℓ. Therefore $|G|$ is at most $7 + 34 \cdot 4 + 12$, or 155. (This counts the seven points on ℓ, the four points other than a on the other lines through a, and an additional point on twelve or fewer of the lines through a.)

Assume there is another 7-point line through a. Therefore, the contraction G/x for any point x in $\ell - \{a\}$ contains a 7-point line through a. Hence, by Lemma 4.1, at most three planes through ℓ have 6- or 7-point lines through a. Hence there are at most seven 6- or 7-point lines through a. Therefore $|G|$ is at most $35 \cdot 4 + 7 \cdot 2 + 1$, or 155. □

Lemma 5.2. *Let G be a rank-4 geometry in $\mathcal{U}(6)$ with no 7-point lines but with a point a on seven coplanar lines. Then $|G| \leq 153$.*

Proof. Assume the plane P contains seven lines through a. Let x be in $P - \{a\}$. Since the contraction G/a contains a 7-point line through x, Lemma 4.1 implies that 6-point lines through x can occur in at most three of the planes through the line $x \vee a$, with P being one of these planes. Since each plane contains at most six 6-point lines through a point, the number of 6-point lines through x is at most 18.

First assume there is a point x in P such that the contraction G/x contains no 7-point lines. It follows from Lemma 3.3 that x is on at most 33 lines in G. Therefore $|G|$ is at most $33 \cdot 4 + 18 + 1$, or 151.

We may now assume that each point in P is on some set of seven coplanar lines. Let z be a point in $P - \{a\}$ and let x be a point in P not on the line $a \vee z$. Let P_1 and P_2 be planes other than P through $a \vee x$ with the property that any 6-point line through x is in one of P, P_1, or P_2. Since the contraction G/z contains a 7-point line, and P_1 and P_2 are restrictions of G/z, it follows from Corollary 2.2 and Lemma 4.1 that each of P_1 and P_2 contains at most three 6-point lines through x. Thus the number of 6-point lines through x is at most $6 + 2 \cdot 3$, so $|G|$ is at most $35 \cdot 4 + 12 + 1$, or 153. □

Having treated rank-4 geometries in $\mathcal{U}(6)$ with either a 7-point line or some point on seven coplanar lines, the one remaining case when the rank is 4 is the following.

Lemma 5.3. *Let G be a rank-4 geometry in $\mathcal{U}(6)$ with no seven coplanar copunctual lines. Then $|G| \leq 156$. Furthermore, $|G|$ is 156 if and only if G is PG$(3, 5)$.*

Proof. Since G contains no seven coplanar copunctual lines, G has no 7-point lines. If there are no 6-point lines, then $|G|$ is at most $35 \cdot 4 + 1$, or 141. Thus we may assume G has at least one 6-point line, say ℓ. Since G has no 7-point lines and no seven coplanar copunctual lines, each plane in G is in $\mathcal{U}(5)$, and hence has at most 31 points. If each plane containing ℓ has at most 27 points, then $|G|$ is at most $7(27-6)+6$, or 153. Thus we may assume that ℓ is in a plane P with at least 28 points. Let x be any point in $G-P$. By Corollary 2.6, x is on at most 31 lines, so $|G|$ is at most $31 \cdot 5+1$, or 156.

Having established the bound, we analyze the case of equality. Since $7(27-6)+6$ is 153, we have that some plane P through ℓ has at least 28 points. By Corollary 2.6, we can have $|G|=156$ only if for each point x in $G-P$, x is on 31 lines and all lines through x are 6-point lines. From this and the fact that all planes of G are in $\mathcal{U}(5)$, it follows that for any such point x, the plane $x \vee \ell$ has 31 points. Thus all planes through ℓ, except perhaps P, have 31 points. Since this argument can now be made using any of these 31-point planes through ℓ, we deduce that all lines in G are 6-point lines.

Since all lines in G are 6-point lines and all planes in G are in $\mathcal{U}(5)$, it follows that all planes in G have 31 points. Therefore each line of G is on six planes. It follows from this and the scum theorem that G is in $\mathcal{U}(5)$. Theorem 1.1 with $q=5$ implies that G is PG$(3,5)$. □

Having established the rank-4 case, we now give the inductive step.

Lemma 5.4. *For $n \geq 5$, rank-n geometries in $\mathcal{U}(6)$ have $(5^n-1)/4$ or fewer points. Furthermore, PG$(n-1,5)$ is the only rank-n geometry in $\mathcal{U}(6)$ with $(5^n-1)/4$ points.*

Proof. Because the proof involves examining the planes through a given line, we need to establish the case $n=5$ separately before we can treat the general case. Let G be a rank-5 geometry in $\mathcal{U}(6)$. Assume G has a 7-point line ℓ. Consider a point x in ℓ. Each 6- or 7-point line ℓ' not equal to ℓ through x determines a plane $\ell \vee \ell'$ with ℓ. There are at most 35 planes through ℓ. By Lemma 4.1, each of these planes contains at most two 6- or 7-point lines through x in addition to ℓ. Therefore the maximum number of 6- or 7-point lines through x is $35 \cdot 2+1$, or 71. Since there are at most $(5^4-1)/4$ lines through x and most contain five or fewer points, $|G|$ is at most

$$\tfrac{1}{4}(5^4-1)4+71 \cdot 2+1=767.$$

This is strictly less than $(5^5-1)/4$, or 781. Now assume G has no 7-point lines, and let x be a point of G. Since x is on at most $(5^4-1)/4$ lines, $|G|$ is at most

$$\tfrac{1}{4}(5^4-1)5+1=\tfrac{1}{4}(5^5-1).$$

Now assume equality holds. It follows that all lines in G are 6-point lines. Therefore each copoint of G has at least $(5^4-1)/4$ points. This and the upper bound in the rank-4 case imply that each copoint has $(5^4-1)/4$ points and is isomorphic PG$(3,5)$. Therefore the colines of G have $(5^3-1)/4$ points. Thus there are

$$\frac{\frac{1}{4}(5^5-1)-\frac{1}{4}(5^3-1)}{\frac{1}{4}(5^4-1)-\frac{1}{4}(5^3-1)}=6$$

copoints over each coline. Therefore by the scum theorem, G is in $\mathcal{U}(5)$. Theorem 1.1 with $q=5$ implies that G is PG$(4,5)$.

Having established the case $n=5$, we can proceed to the general case. Assume $n>5$ and that the result holds for ranks $n-1$ and $n-2$. Let G be a rank-n geometry in $\mathcal{U}(6)$. Assume G has a 7-point line ℓ. Consider a point x in ℓ. There are at most $(5^{n-2}-1)/4$ planes through ℓ, and hence at most

$$\tfrac{1}{4}(5^{n-2}-1)2+1$$

6- or 7-point lines through x. There are at most $(5^{n-1}-1)/4$ lines through x. Therefore $|G|$ is at most

$$\tfrac{1}{4}(5^{n-1}-1)4+\Big(\tfrac{1}{4}(5^{n-2}-1)2+1\Big)2+1.$$

This is strictly less than $(5^n-1)/4$. Now assume G has no 7-point lines, and let x be a point of G. Since x is on at most $(5^{n-1}-1)/4$ lines, $|G|$ is at most

$$\tfrac{1}{4}(5^{n-1}-1)5+1=\tfrac{1}{4}(5^n-1).$$

Assume equality holds. It follows that all lines in G are 6-point lines, so each copoint of G has at least $(5^{n-1}-1)/4$ points. Therefore each copoint has $(5^{n-1}-1)/4$ points and is isomorphic to PG$(n-2,5)$. The colines of G have $(5^{n-2}-1)/4$ points. Thus there are

$$\frac{\frac{1}{4}(5^n-1)-\frac{1}{4}(5^{n-2}-1)}{\frac{1}{4}(5^{n-1}-1)-\frac{1}{4}(5^{n-2}-1)}=6$$

copoints over each coline. As in the rank-5 case, the scum theorem and Theorem 1.1 with $q=5$ imply that G is PG$(n-1,5)$. □

6. An alternative proof.

A major step in the proof of Theorem 1.2 is to prove the upper bound in the rank-4 case. In this section, we sketch an alternative proof of the rank-4 upper bound, that if G is a rank-4 geometry in $\mathcal{U}(6)$, then $|G|\le 156$.

We begin the two-part proof with the following lemma.

Lemma 6.1. *Let G be a rank-4 geometry in $\mathcal{U}(6)$ such that for each plane P of G, there is a point x in the complement $G-P$ on no 7-point lines. Then $|G|\le 156$.*

Proof. Let a be a point in G on no 7-point lines. Since $35\cdot 4+15+1$ is 156, we may assume that there are at least sixteen 6-point lines on a. From this, one can prove that for some 6-point line ℓ on a, at least two planes through ℓ each contain three or more additional 6-point lines on a. Note that ℓ is modular in each plane P containing ℓ. (Otherwise, there exists a line in P not intersecting ℓ; contracting by a point b on that line projects a seventh point onto ℓ and so the contraction G/b has a 7-point line through a and at least three other 6- or 7-point lines through a, contradicting Lemma 4.1.)

Since ℓ is modular in P, each point c in $P-\ell$ is on precisely six lines in P. If there is a point c in $P-\ell$ on no 6-point line, then $|P|\le 6\cdot 4+1$, and hence, $|P|\le 25$. Therefore we may assume that each point in $P-\ell$ is on a 6-point line. To show that P is in $\mathcal{U}(5)$, it suffices to show that each point d in ℓ is on at most six lines in P. Suppose that a point d in ℓ is on seven lines in P. If there is a 6-point line ℓ' in P not containing d, then ℓ' intersects ℓ at a point not equal to d and the six points on ℓ' determine six lines through d of the form $d\vee e$, where e is a point on ℓ'. Let f be a point on the seventh line through d not equal to d. Then there are seven lines through f, contradicting the fact that ℓ is modular. Thus, we may assume that all 6-point lines in P contain d. Since all points in $P-\ell$ are on 6-point lines, all seven lines through d are 6-point lines and there are 36 points in P, contradicting Lemma 3.1. We conclude that every point in ℓ is on at most six lines in P.

To finish the proof, observe that if each of the planes containing ℓ has at most 27 points, then $|G|$ is at most $7(27-6)+6$, or 153. On the other hand, if ℓ is in a plane P with at least 28 points, then that plane P is in $\mathcal{U}(5)$. By hypothesis, there is a point x in $G-P$ on no 7-point lines. Hence, by Corollary 2.6, x is on at most 31 lines, and so $|G|$ is at most $31\cdot 5+1$, or 156. □

The second part of the proof deals with the cases not covered by Lemma 6.1.

Lemma 6.2. *Let G be a rank-4 geometry in $\mathcal{U}(6)$. Suppose that there is a plane P in G such that each point x in the complement $G-P$ is on at least one 7-point line of G. Then $|G|\le 155$.*

Proof. Begin with the special case in which all 7-point lines not in P contain some fixed point a. If a is in P, then by Corollary 4.2, $|G-P|\le 42$ and hence $|G|\le 77$. If a is not in P, then each point x of $G-P$ is on a 7-point line through a and hence $|G|$ is at most $7\cdot 6+(35-7)+1$, or 71. We may therefore assume that each point of G is disjoint from at least one 7-point line. By Corollary 2.3, we may assume that there are no 6-point lines in G. Consider the lines through any point a. By Corollary 4.2, at most seven of those lines have seven points. The rest have five or fewer points. From this, we conclude that $|G|$ is at most $35\cdot 4+7\cdot 2+1$, or 155. □

With the rank-4 upper bound just obtained, we can complete the proof of the upper bound in Theorem 1.2 using the argument in the proof of Lemma 5.4.

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