Semidirect Sums of Matroids

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These slides are available at
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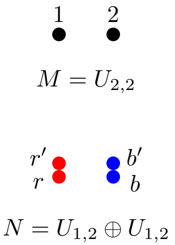
Fixed notation: matroids M on S, N on T, with $S \cap T = \emptyset$.

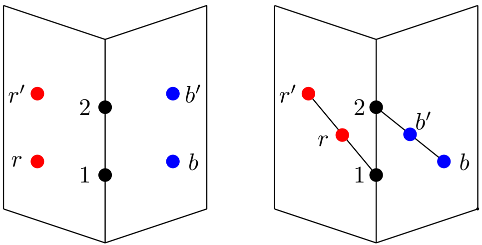
Definition

A matroid G on $S \cup T$ is a semidirect sum of M and N if $G|S = M$ and $G/S = N$.

E.g., $M \oplus N$.

The order of M and N matters.





Let D_M (resp., D_N) be a matrix representation of M (resp., N) over a field F.
The column matroid of any matrix of the form

$$\begin{pmatrix} D_M & U \\ 0 & D_N \end{pmatrix},$$

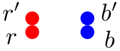
for any matrix U over F of the appropriate size, is a semidirect sum of M and N.

Conversely, semidirect sums of M and N that are representable over F have representations of this form.

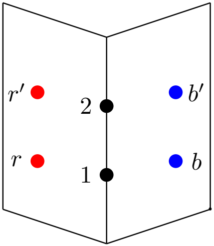
The free product, $M \square N$, defined by H. Crapo and W. Schmitt, is the freest semidirect sum of M and N.

Indeed, G is a semidirect sum of M and N if and only if $M \oplus N \le G \le M \square N$ in the weak order.

$M = U_{2,2}$



$N = U_{1,2} \oplus U_{1,2}$



$M \square N$

The independent sets of $M \square N$ are the sets $X \cup Y$ with X independent in M, $Y \subseteq T$, and $r(M) - |X| \ge |Y| - r_N(Y)$.

Each matroid G on a set E, with $|E| > 1$, is a semidirect sum:
if $\emptyset \subsetneq X \subsetneq E$, then G is a semidirect sum of $G|X$ and G/X.

If G is a semidirect sum of M and N, then
- G^* is a semidirect sum of N^* and M^*, and
- $r(G) = r(M) + r(N)$.

For matroids G and H on the same set E, their union, $G \vee H$, is the matroid on E whose collection of independent sets is

$$\mathcal{I}(G \vee H) = \{I \cup J \,:\, I \in \mathcal{I}(G),\, J \in \mathcal{I}(H)\}.$$

E.g., $U_{r,n} \vee U_{r',n} = U_{\min\{r+r',n\},n}$.

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For $X \subseteq E$, $\quad (G \vee H)\backslash X = (G\backslash X) \vee (H\backslash X)$.

Compare: $\quad (U_{2,4} \vee U_{2,4})/x = U_{4,4}/x = U_{3,3}$
versus $\quad (U_{2,4}/x) \vee (U_{2,4}/x) = U_{1,3} \vee U_{1,3} = U_{2,3}$.

If each $x \in X$ is a loop of at least one of G and H, then

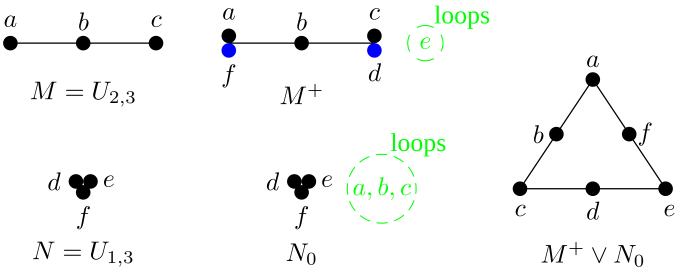
$$(G \vee H)/X = (G/X) \vee (H/X).$$

Proof: If x is a loop of H but not of G, then TFAE

- $I \in \mathcal{I}((G \vee H)/x)$
- $I \cup x \in \mathcal{I}(G \vee H)$
- $I \cup x = (I_G \cup x) \cup I_H$ for some $I_G \cup x \in \mathcal{I}(G)$ and $I_H \in \mathcal{I}(H)$
- $I = I_G \cup I_H$ for some $I_G \in \mathcal{I}(G/x)$ and $I_H \in \mathcal{I}(H/x)$
- $I \in \mathcal{I}((G/x) \vee (H/x))$.

Theorem

If M^+ is an extension of M to $S \cup T$ with $r(M) = r(M^+)$, and $N_0 = N \oplus U_{0,S}$, where $U_{r,S}$ is the rank-r matroid on S, then $M^+ \vee N_0$ is a semidirect sum of M and N.



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Proof.

$$(M^+ \vee N_0)|S = (M^+|S) \vee (N_0|S) = M.$$

Each $x \in S$ is a loop of N_0, so

$$(M^+ \vee N_0)/S = (M^+/S) \vee (N_0/S) = N. \qquad \square$$

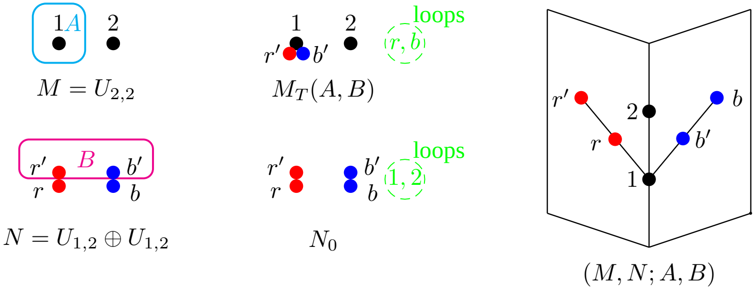
Fix $A \subseteq S$ and $B \subseteq T$.

Extend M to $M_T(A, B)$ on $S \cup T$ by

- adding B freely to $\mathrm{cl}_M(A)$ and
- making elements in $T - B$ loops.

Definition

The principal sum $(M, N; A, B)$ is $M_T(A, B) \vee N_0$.



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Definition

The principal sum $(M, N; A, B)$ is $M_T(A, B) \vee N_0$.

E.g., $(M, N, \emptyset, \emptyset) = M \oplus N$.
Also, $(M, N, \emptyset, B) = (M, N, A, \emptyset) = M \oplus N$ for any A and B.

Lemma

The independent sets of $M_T(A, B)$ are the sets $I \cup J$ with

$$I \in \mathcal{I}(M), \qquad J \subseteq B, \qquad \textit{and} \qquad r_M(I \cup A) - |I| \geq |J|.$$

Lemma

The independent sets of $M_T(A, B)$ are the sets $I \cup J$ with

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Theorem

The independent sets of $(M, N; A, B)$ are the unions of three disjoint sets, I, D, and D', where

$$I \in \mathcal{I}(M), \qquad D \in \mathcal{I}(N), \qquad D' \subseteq B, \quad \& \quad |D'| \leq r_M(I \cup A) - |I|.$$

Independent Sets — II

The independent sets of $(M, N; A, B)$ are the unions of three disjoint sets, I, D, and D', where

$$I \in \mathcal{I}(M), \qquad D \in \mathcal{I}(N), \quad D' \subseteq B, \quad \& \quad |D'| \leq r_M(I \cup A) - |I|.$$

The independent sets of $(M, N; S, T)$ are the unions of disjoint triples $I \in \mathcal{I}(M)$, $D \in \mathcal{I}(N)$, and $D' \subseteq T$ with

$|D'| \leq r_M(I \cup S) - |I|$, that is,

$|D \cup D'| - r_N(D) \leq r(M) - |I|$.

The independent sets of $M \square N$ are the sets $X \cup Y$ with $X \in \mathcal{I}(M)$, $Y \subseteq T$, and $|Y| - r_N(Y) \leq r(M) - |X|$.

Thus, $(M, N; S, T) = M \square N$.

Theorem

Let $P = (M, N; A, B)$. For $X \subseteq S$ and $Y \subseteq T$, $r_P(X \cup Y)$ is the minimum of

$$r_M(X \cup A) + r_N(Y) \quad \& \quad r_M(X) + r_N(Y - B) + |Y \cap B|.$$

Idea of the proof:

We want the largest $I \cup D \cup D'$ with

$$I \in \mathcal{I}(M), \quad D \in \mathcal{I}(N), \quad D' \subseteq B, \quad \& \quad |D'| \leq r_M(I \cup A) - |I|.$$

Let I be a basis of $M|X$.

Let D be a basis of $N|Y$ that has minimal intersection with B.

Then D' is either $(Y \cap B) - D$ or a $(r_M(I \cup A) - |I|)$-subset of it.

An Alternative View

Set $G_0 = (M/A) \oplus U_{0,A} \oplus N$ and $G_1 = M \oplus (N \backslash B) \oplus U_{|B|,B}$.

G_0 is a quotient of $M \oplus N$, which is a quotient of G_1.

$r_M(X \cup A) + r_N(Y) = r_{G_0}(X \cup Y) + r_M(A)$.

$r_M(X) + r_N(Y - B) + |Y \cap B| = r_{G_1}(X \cup Y)$.

Thus, $(M, N; A, B)$ is the $r_M(A)$-th Higgs lift of G_0 toward G_1.

Higgs Lifts Yield More Semidirect Sums

Let M' be a quotient of M with $r(M) - r(M') = i$.

Let N' be a lift of N.

Set $G_0 = M' \oplus N$ and $G_1 = M \oplus N'$.

Theorem

G_0 is a quotient of G_1.

The i-th Higgs lift of G_0 up to G_1 is a semidirect sum of M and N.

Neither the union nor the Higgs lift construction encompasses the other.

Theorem

The dual: $(M, N; A, B)^* = (N^*, M^*; B, A)$.

H. Crapo and W. Schmitt proved $(M \mathbin{\square} N)^* = N^* \mathbin{\square} M^*$.

Fails for $M^+ \vee N_0$ in general.

Corollary

If $cl_M(A) = cl_M(A')$ *and* $cl_{N^*}(B) = cl_{N^*}(B')$, *then* $(M, N; A, B) = (M, N; A', B')$.

The converse holds except for $M \oplus N$.

Corollary

For $B \subseteq T$, *if* B_0 *is a basis of* $N^*|cl_{N^*}(B)$ *and if* $M_T(A, B_0) \leq M' \leq M_T(A, cl_{N^*}(B))$ *(weak order), then* $M' \vee N_0 = (M, N; A, B)$.

Other Basic Concepts

Using the rank function or the theory of Higgs lifts, we can identify
the closure operator,
flats,
cyclic flats (flats that are unions of circuits),
circuits . . .

Theorem

$P = (M, N; A, B)$ is disconnected if and only if

1. $P = M \oplus N$, *or*
2. *either M has loops or N has coloops, or*
3. $A \neq \emptyset$ *and M has a separator X with* $A \subseteq X \subsetneq S$, *or*
4. $B \neq \emptyset$ *and N has a nonempty separator Y with* $Y \cap B = \emptyset$.

Theorem

Let M, N, and K be matroids on disjoint sets S, T, and U. If $A \subseteq S$, $B \subseteq T$, and $C \subseteq U$, then

$$((M, N; A, B), K; A \cup B, C) = (M, (N, K; B, C); A, B \cup C).$$

Cyclic sets are unions of circuits.
$\mathcal{Z}(G)$ is the set of cyclic flats of G.

Theorem

Let $P = (M, N; A, B)$.

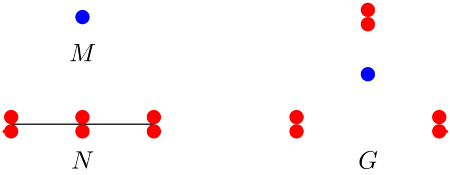
- *For* $A \in \mathcal{Z}(M)$, *if* M *and* N *are transversal, then so is* P.
- *For* $A \in \mathcal{Z}(M)$ *and* $B \in \mathcal{Z}(N^*)$, *if* M *and* N *are fundamental transversal, then so is* P.

Theorem

Let G be a semidirect sum of M and N of the form $M^+ \vee N_0$.

- *If G is transversal, then so are M and N.*
- *If G is fundamental transversal, then so are M and N.*

The same is not true of the Higgs lift construction.



Open Problem

Which matroids are principal sums?