STRONGLY INEQUIVALENT REPRESENTATIONS AND TUTTE POLYNOMIALS OF MATROIDS

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ABSTRACT. We develop constructive techniques to show that non-isomorphic 3-connected matroids that are representable over a fixed finite field and that have the same Tutte polynomial abound. In particular, for most prime powers q, we construct infinite families of sets of 3-connected matroids for which the matroids in a given set are non-isomorphic, are representable over $\mathrm{GF}(q)$, and have the same Tutte polynomial. Furthermore, the cardinalities of the sets of matroids in a given family grow exponentially as a function of rank, and there are many such families.

In Memory of Gian-Carlo Rota

1. INTRODUCTION

The Tutte polynomial is of central importance in matroid theory and has applications in many areas of mathematics (see [8]). Certain matroids are distinguished from all other matroids by their Tutte polynomials. Examples of such matroids include the following: uniform matroids; cycle matroids of complete graphs [4]; Dowling lattices of rank four or more based on groups of prime order [4], as well as the jointless counterparts of these geometries [16], and their complete principal truncations by modular flats [15]; finite projective and affine geometries of rank four or more [4], and, more generally, $\mathrm{PG}(n-1,q)\backslash\mathrm{PG}(k-1,q)$ for integers n and k with $n \geq 4$ and $n \geq k$ (see [1]). On the other hand, non-isomorphic matroids may have the same Tutte polynomial. The following proposition is a more precise formulation of this assertion.

Proposition 1.1. *For any integer k, there are at least k non-isomorphic simple matroids that have the same Tutte polynomial.*

In this paper, we give a constructive proof of a stronger assertion. Before stating the stronger result, we mention three proofs of Proposition 1.1 to highlight what is gained with the techniques we develop.

A non-constructive proof of Proposition 1.1 proceeds as follows [8, Exercise 6.9]. Knuth [11] proved that there are at least

$$\frac{1}{m!}2^{\left\lceil \binom{m}{\lfloor m/2 \rfloor}/2m \right\rceil}$$

non-isomorphic simple matroids on m elements. One can also show that there are at most

$$2^{(m+1)^3/4}$$

distinct Tutte polynomials arising from matroids on m elements. From these results, one can deduce Proposition 1.1.

For a constructive proof of Proposition 1.1, one could use the following three results: Dowling lattices based on non-isomorphic groups are not isomorphic [9, Theorem 8]; all Dowling lattices of the same rank based on groups of the same order have the same Tutte polynomial [4, Proposition 4.1]; for any integer k, there is an integer t such that there are at least k non-isomorphic groups of order t. However, Dowling lattices based on non-cyclic finite groups are not representable over any field [9, Theorem 9], so at most one of the Dowling lattices arising from this construction is representable over a field.

A third proof of Proposition 1.1 uses the following basic result: the Tutte polynomial of a direct sum of matroids is the product of the Tutte polynomials of the matroids. Let M and N be two non-isomorphic matroids that have the same Tutte polynomial (see the beginning of Section 2 for such matroids). For i with $0 \leq i \leq k-1$, let M_i be the direct sum of i copies of M and $k-1-i$ copies of N. It follows that the k matroids $M_0, M_1, \ldots, M_{k-1}$ are non-isomorphic and have the same Tutte polynomial.

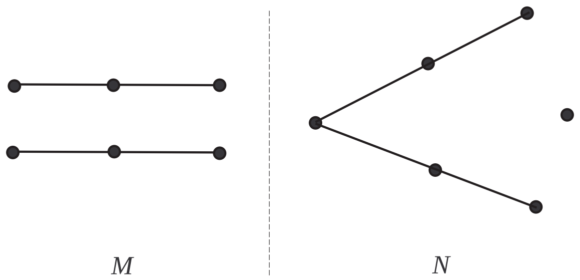

FIGURE 1. Two matroids that have the same Tutte polynomial.

Theorem 1.2 improves on Proposition 1.1 in that the matroids are required to be 3-connected and representable over a fixed finite field.

Theorem 1.2. *For any integer k and any prime power q with $q \geq 7$, there are at least k non-isomorphic 3-connected matroids that are representable over $\mathrm{GF}(q)$ and that have the same Tutte polynomial.*

Our proof of Theorem 1.2 is constructive and rests on strongly inequivalent representations of matroids, which we introduce in Section 4. In particular, we show that deleting, from the projective geometry $\mathrm{PG}(n-1,q)$, the images of inequivalent representations of certain simple matroids yields non-isomorphic simple matroids that have the same Tutte polynomial. This allows us to construct many large sets of non-isomorphic 3-connected matroids that are representable over a fixed finite field and that have the same Tutte polynomial. Indeed, for most prime powers q, we produce families of such sets, indexed by rank, where the cardinalities of the sets in each family grow exponentially as a function of rank.

Section 2 reviews the background we require on Tutte polynomials. Basic ideas about inequivalent representations of matroids as well as key examples from [13] are covered in Section 3. Section 4 gives the main results: a method for deducing that certain deletions of projective geometries (that are known to have the same Tutte polynomial) are not isomorphic, and, building on this result and examples from [13], constructions of large sets of simple matroids that are representable over a fixed finite field and that have the same Tutte polynomial. Several directions for extending these results are discussed in the concluding section of the paper.

All notation and terminology for matroid theory that is not defined here can be found in [12]. We refer to simple matroids as *geometries* (short for combinatorial geometries). An *embedding* of a matroid M, on the ground set S, into a matroid N, on the ground set T, is an injection $\phi : S \to T$ such that the map $\phi : S \to \phi(S)$ is an isomorphism of M onto the restriction $N|\phi(S)$. Embeddings of a geometry M in $\mathrm{PG}(n-1,q)$ are also called *representations of M over* $\mathrm{GF}(q)$. Relative to a coordinatization of the rank-n projective geometry $\mathrm{PG}(n-1,q)$ over the finite field $\mathrm{GF}(q)$, a representation of a geometry M on the ground set S can be specified by listing the images of the elements of S in the chosen coordinatization; these images are customarily listed as the columns of a matrix over $\mathrm{GF}(q)$. We shift freely between such concrete matrix representations of geometries and the more abstract, coordinate-free perspective. Since we focus mainly on geometries, we often use the term point (i.e., flat of rank 1) to refer to an element of the ground set of a geometry.

2. TUTTE POLYNOMIALS

The *Tutte polynomial* $t(M;x,y)$ of a matroid M on the ground set S is defined as follows.

$$t(M;x,y) = \sum_{A \subseteq S} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}$$

Our goal is to produce many non-isomorphic geometries, all representable over the same finite field, that have the same Tutte polynomial. The following rank-3 geometries provide simple examples of such geometries. Fix two integers m and n, both of which exceed two. (The case $m = n = 3$ is illustrated in Figure 1.) Let M be the geometry of rank 3 that consists of two disjoint lines, one of which contains m points, the other of which contains n points. Let N be the geometry formed as follows: take the parallel connection of the uniform matroids $U_{2,m}$ and $U_{2,n}$, and add a point freely to the resulting rank-3 geometry. Both M and N are representable over $\mathrm{GF}(q)$ for sufficiently large prime powers q; in particular, the geometries shown in Figure 1 are representable over $\mathrm{GF}(q)$ for all prime powers q that exceed three. The geometries M and N have the following Tutte polynomial.

$$(x-1)^3+(m+n)(x-1)^2+\binom{m+n}{2}(x-1)+\sum_{i=3}^{\max(m,n)}\left(\binom{m}{i}+\binom{n}{i}\right)(x-1)(y-1)^{i-2}$$
$$+\sum_{i=3}^{m+n}\left(\binom{m+n}{i}-\binom{m}{i}-\binom{n}{i}\right)(y-1)^{i-3}$$

In contrast to such isolated examples, the cardinalities of the sets of geometries we construct in Theorem 4.4, with geometries in the same set being non-isomorphic and having the same Tutte polynomial, grow exponentially in the rank.

The following theorem from [6] plays an important role in Section 4. (The statement below follows from, but does not capture the full power of, Proposition 5.9 in [6].)

Proposition 2.1. *Assume that M and N are geometries that are representable over $\mathrm{GF}(q)$ and that have the same Tutte polynomial. Let S and T be subsets of the ground set of $\mathrm{PG}(n-1,q)$ so that the restrictions $\mathrm{PG}(n-1,q)|S$ and $\mathrm{PG}(n-1,q)|T$ are isomorphic to M and N, respectively. Then the deletions $\mathrm{PG}(n-1,q)\backslash S$ and $\mathrm{PG}(n-1,q)\backslash T$ have the same Tutte polynomial.*

To illustrate this result, note that the two matroids M and N shown in Figure 1 are representable over all finite fields $\mathrm{GF}(q)$ with q a prime power that exceeds three. Since M and N have the same Tutte polynomial, it follows from Proposition 2.1 that if $\mathrm{PG}(n-1,q)|S$ and $\mathrm{PG}(n-1,q)|T$ are isomorphic to M and N, respectively, then the deletions $\mathrm{PG}(n-1,q)\backslash S$ and $\mathrm{PG}(n-1,q)\backslash T$ have the same Tutte polynomial. Note that $\mathrm{PG}(n-1,q)\backslash S$ and $\mathrm{PG}(n-1,q)\backslash T$ are not isomorphic since the two $(q-2)$-point lines of $\mathrm{PG}(n-1,q)\backslash S$ intersect in a point while those of $\mathrm{PG}(n-1,q)\backslash T$ do not. We conclude that for every prime power q that exceeds three and every integer n with $n \geq 3$, there are at least two non-isomorphic six-element deletions of $\mathrm{PG}(n-1,q)$ that have the same Tutte polynomial.

As illustrated in the preceding example, when drawing further conclusions from Proposition 2.1, the issue of isomorphism is important. In general, no conclusion about whether $\mathrm{PG}(n-1,q)\backslash S$ and $\mathrm{PG}(n-1,q)\backslash T$ are isomorphic can be drawn from knowing whether $\mathrm{PG}(n-1,q)|S$ and $\mathrm{PG}(n-1,q)|T$ are isomorphic.

Note that we can prove Theorem 1.2 by letting S_i be a subset of the ground set of $\mathrm{PG}(n-1,q)$ for which $\mathrm{PG}(n-1,q)|S_i$ is isomorphic to the direct sum of i copies of M and $k-1-i$ copies of N where M and N are the geometries of Figure 1; the k geometries $\mathrm{PG}(n-1,q)\backslash S_i$, for i with $0 \leq i \leq k-1$, are 3-connected, non-isomorphic, and have the same Tutte polynomial. However, to have k such geometries, n must be at least $3(k-1)$. The techniques developed in Section 4 improve on this considerably. On the other hand, since M and N are representable over $\mathrm{GF}(q)$ for all prime powers q with $q \geq 4$, we get the following slight strengthening of Theorem 1.2.

Theorem 2.2. *For any integer k, any prime power q with $q \geq 4$, and any integer n with $n \geq 3(k-1)$, there are at least k non-isomorphic 3-connected geometries of rank n that are representable over $\mathrm{GF}(q)$ and that have the same Tutte polynomial.*

3. Inequivalent representations of geometries

Let M be a geometry that is representable over the finite field $\mathrm{GF}(q)$. Let ϕ be an embedding, or representation, of M in $\mathrm{PG}(n-1,q)$ and let σ be an automorphism of $\mathrm{PG}(n-1,q)$. (Such automorphisms are induced by bijective semilinear transformations on the associated vector space [2, Theorem 2.26].) Note that the composition $\sigma\phi$ is also a representation of M in $\mathrm{PG}(n-1,q)$. Two representations ϕ and ψ of M are *equivalent* if there is an

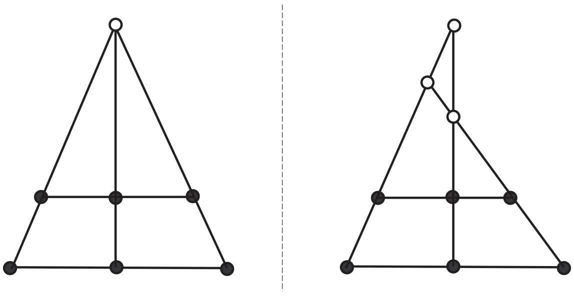

FIGURE 2. One possible cause of inequivalent representations.

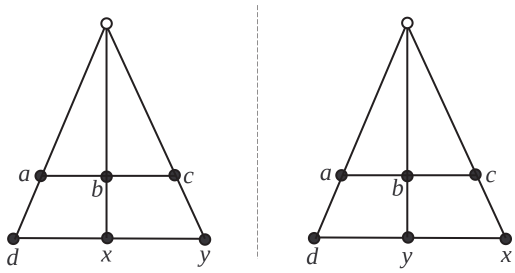

FIGURE 3. Automorphisms of the geometry can lead to inequivalent representations.

automorphism σ of $\mathrm{PG}(n-1,q)$ such that $\psi = \sigma\phi$; if there is no such σ, the representations are *inequivalent*. (The literature contains two notions of equivalent representations for matroids of rank two; this is not an issue here since all matroids we consider have rank three or more.) A geometry M of rank n is *uniquely representable over* $\mathrm{GF}(q)$ if M is representable over $\mathrm{GF}(q)$ and all representations of M in $\mathrm{PG}(n-1,q)$ are equivalent.

The geometry M of Figure 1 has inequivalent representations over $\mathrm{GF}(q)$ for sufficiently large q. In particular, for sufficiently large q there are embeddings of M in $\mathrm{PG}(2,q)$ so that some triple of disjoint two-point lines of M, when viewed in $\mathrm{PG}(2,q)$, intersect in a point of $\mathrm{PG}(2,q)$, as well as embeddings so that no triple of disjoint two-point lines of M has this property. This situation is suggested in Figure 2. Representations can also be inequivalent due to issues of labeling, that is, non-trivial automorphisms; this is suggested in Figure 3.

It is immediate that, as the name suggests, equivalence of representations is an equivalence relation on the set of representations of a geometry. Obviously, only representations in projective geometries of the same rank can be equivalent. Often one considers equivalence only on representations of geometries of rank n in projective geometries of rank n. Since we will need to consider equivalence on representations of geometries of rank n in projective geometries of rank n or greater, we show in Theorem 3.2 that this yields no essential difference. Indeed, for any geometry M of rank n and any integer m with $m \geq n$, there are precisely as many equivalence classes of representations of M in $\mathrm{PG}(n-1,q)$ as in $\mathrm{PG}(m-1,q)$. This justifies the following terminology: for a geometry M of rank n, the number of inequivalent representations of M over $\mathrm{GF}(q)$ refers to the number of equivalence classes of representations of M in the projective geometry $\mathrm{PG}(m-1,q)$ for any fixed integer m with $m \geq n$.

We start by showing that projective geometries are uniquely representable in projective geometries of higher rank over the same field.

Lemma 3.1. *For any prime power q and any integers m and n with $m \geq n \geq 3$, the geometry $\mathrm{PG}(n-1,q)$ is uniquely representable in $\mathrm{PG}(m-1,q)$.*

Proof. Let N be an extension of $\mathrm{PG}(n-1,q)$ that is isomorphic to $\mathrm{PG}(m-1,q)$. Consider two embeddings, ϕ and ψ, of $\mathrm{PG}(n-1,q)$ into $\mathrm{PG}(m-1,q)$. By extending the injective semilinear transformations, defined on the underlying vector spaces, that give rise to ϕ and ψ, it is easy to see that ϕ and ψ can be extended to isomorphisms Φ and Ψ of N onto $\mathrm{PG}(m-1,q)$. Therefore $\Psi\Phi^{-1}$ is an automorphism of $\mathrm{PG}(m-1,q)$ and $\psi = (\Psi\Phi^{-1})\phi$, as needed. □

Theorem 3.2. *Assume n is an integer with $n \geq 3$. Let M be a geometry of rank n that is representable over $\mathrm{GF}(q)$ and let ϕ_1 and ϕ_2 be embeddings of M in $\mathrm{PG}(n-1,q)$. Let m be any integer with $m \geq n$ and let τ_1 and τ_2 be any embeddings of $\mathrm{PG}(n-1,q)$ in $\mathrm{PG}(m-1,q)$. The representations ϕ_1 and ϕ_2 are equivalent if and only if the representations $\tau_1\phi_1$ and $\tau_2\phi_2$ are equivalent.*

Proof. Assume that ϕ_1 and ϕ_2 are equivalent, say $\phi_1 = \sigma'\phi_2$ for an automorphism σ' of $\mathrm{PG}(n-1,q)$. Both $\tau_1\sigma'$ and τ_2 are embeddings of $\mathrm{PG}(n-1,q)$ in $\mathrm{PG}(m-1,q)$, so by Lemma 3.1, there is an automorphism σ of $\mathrm{PG}(m-1,q)$ such that $\tau_1\sigma' = \sigma\tau_2$. By composing these functions with ϕ_2 and using the equality $\phi_1 = \sigma'\phi_2$, we get $\tau_1\phi_1 = \tau_1\sigma'\phi_2 = \sigma\tau_2\phi_2$, so $\tau_1\phi_1$ and $\tau_2\phi_2$ are equivalent representations of M.

Now assume that $\tau_1\phi_1$ and $\tau_2\phi_2$ are equivalent, so $\tau_1\phi_1 = \sigma\tau_2\phi_2$ for some automorphism σ of $\mathrm{PG}(m-1,q)$. Let S be the ground set of $\mathrm{PG}(n-1,q)$. Since M has rank n, it follows that $\tau_1(S) = \sigma\tau_2(S)$, so for the inverse τ_1^{-1} of the restriction $\tau_1 : S \to \tau_1(S)$, the map $\tau_1^{-1}\sigma\tau_2$ is an automorphism of $\mathrm{PG}(n-1,q)$. From this and the equality $\tau_1\phi_1 = \sigma\tau_2\phi_2$, we get $\phi_1 = \tau_1^{-1}\sigma\tau_2\phi_2$, so ϕ_1 and ϕ_2 are equivalent representations of M. □

For a prime power q, let $b(q)$ be the smallest integer (if such an integer exists) so that each 3-connected geometry that is representable over $\mathrm{GF}(q)$ has at most $b(q)$ inequivalent representations over $\mathrm{GF}(q)$. Brylawski and Lucas [7] showed that $b(2)$ and $b(3)$ are 1 even without the hypothesis of 3-connectivity. Kahn [10] showed that $b(4)$ is also 1; the assumption of 3-connectivity is necessary in this result. It is well known that these results play crucial roles in the proofs of the excluded-minor characterizations for representability over $\mathrm{GF}(2)$, $\mathrm{GF}(3)$, and $\mathrm{GF}(4)$. Kahn also showed that for all prime powers q with $q > 4$, the integer $b(q)$, if it exists, is at least two; this is recognized as a major obstacle to finding excluded-minor characterizations for representability over $\mathrm{GF}(q)$ for such prime powers q.

Kahn conjectured that $b(q)$ exists for all prime powers q. Oxley, Vertigan, and Whittle [13] proved that Kahn's conjecture is true for $q = 5$ and false for all larger prime powers q. Oxley, Vertigan, and Whittle gave two classes of counterexamples to Kahn's conjecture, M_n and N_n, the n-swirl and the free n-spike; these geometries are a key element in our work in Section 4 and so are reviewed here. Throughout the rest of this paper, M_n and N_n denote the matroids defined below, with the labeling of the elements of the n-swirl and the free n-spike as indicated in the matrix representation for these matroids given below.

Assume that q is a prime power that exceeds five and that is not of the form 2^t where $2^t - 1$ is prime. (Note that for $2^t - 1$ to be prime, the exponent t must be prime; the converse is not true. Recall that primes of the form $2^t - 1$ are called Mersenne primes; it is a longstanding open problem whether there are infinitely many Mersenne primes.) Let A be a proper non-trivial subgroup of $\mathrm{GF}^*(q)$, the group of units of $\mathrm{GF}(q)$. Let n be an integer that exceeds two. Let M_n be the matroid that is represented over $\mathrm{GF}(q)$ by the n by $3n$ matrix

$$\begin{array}{ccccccccccccccc} e_1 & e_2 & \dots & e_n & f_1 & g_1 & f_2 & g_2 & f_3 & g_3 & \dots & f_{n-1} & g_{n-1} & f_n & g_n \end{array}$$
$$\begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \\ 0 & 1 & \dots & 0 & 1 & \alpha_1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \alpha_2 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_3 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & \alpha_{n-1} & \beta_1 & \beta_2 \end{pmatrix}$$

where $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are arbitrary elements of $A - \{1\}$ and β_1, β_2 are arbitrary distinct elements of $\mathrm{GF}^*(q)$ that are not in the coset $(-1)^n A$ of the subgroup A of $\mathrm{GF}^*(q)$. Oxley, Vertigan, and Whittle prove the following proposition. (The statement below is based on Lemma 5.1 of [13] and the results contained in the proof, rather than the statement, of Proposition 5.2 of [13].)

Proposition 3.3. *Let q, A, and M_n be as given above, where q is the t-th power of a prime. The geometry M_n does not depend upon the choice of the elements $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \beta_1, \beta_2$. Furthermore, M_n has at least*

$$\frac{(|A|-1)^{n-1}(q-1-|A|)(q-2-|A|)}{t}$$

inequivalent representations over $\mathrm{GF}(q)$.

Oxley, Vertigan, and Whittle note that the n-swirl, M_n, is the matroid formed from the n-whirl by adding n points, with one point added freely to each three-point line of the n-whirl. Thus, M_n does not even depend on the choice of q.

To define free n-spikes, the second class of counterexamples to Kahn's conjecture given by Oxley, Vertigan, and Whittle, consider a *proper* prime power $q = p^t$ that exceeds five. (By a proper prime power, we mean that q is not prime, that is, $t \geq 2$.) Let A be a proper non-trivial additive subgroup of $\mathrm{GF}(q)$. Let n be an integer that exceeds three. Let N_n be the matroid that is represented over $\mathrm{GF}(q)$ by the n by $2n+1$ matrix

$$\begin{array}{c} \begin{array}{cccccccccccc} e_1 & e_2 & \dots & e_{n-1} & g & f_1 & f_2 & f_3 & \dots & f_{n-1} & e_n & f_n \end{array} \\ \left(\begin{array}{cccccccccccc} 1 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 1 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 1 & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 & \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-1} & \beta_1 & \beta_2 \end{array}\right) \end{array}$$

where $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are any non-zero elements of A and β_1, β_2 are any distinct elements of $\mathrm{GF}(q) - A$. The following proposition is based on Lemma 5.3 of [13] and the proof, rather than the statement, of Proposition 5.4 of [13].

Proposition 3.4. *Let q, A, and N_n be as given above, where q is a proper prime power, say the t-th power, p^t, of a prime p. The geometry N_n does not depend on the choice of the elements $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \beta_1, \beta_2$. Furthermore, N_n has at least*

$$\frac{(|A|-1)^{n-2}(q-|A|)(q-1-|A|)}{t}$$

inequivalent representations over $\mathrm{GF}(q)$.

As is true of the geometry M_n, the geometry N_n does not depend on the field $\mathrm{GF}(q)$. The geometry N_n, the free n-spike, is the freest rank-n geometry that consists of n three-point lines that contain the point g. (See [13, Lemma 5.3].)

Part of why the n-swirl and the free n-spike can have so many inequivalent representations is that they have few points relative to $\mathrm{PG}(n-1,q)$. Indeed, any geometry with "enough" points is uniquely representable, as the following proposition from [3] shows.

Proposition 3.5. *Assume that n is an integer with $n \geq 3$. A geometry of rank n that is representable over* $\mathrm{GF}(q)$ *and that has at least q^{n-1} points is uniquely representable over* $\mathrm{GF}(q)$.

Note that [3] contains two results that are stronger than Proposition 3.5 in the case of odd q. Given the results in the present paper, it may be worth proving even stronger results of this type.

4. Geometries that have the same Tutte polynomial

Assume that ϕ and ψ are representations of a geometry M, on the ground set S, in $\mathrm{PG}(n-1,q)$. If ϕ and ψ are equivalent, then the deletions $\mathrm{PG}(n-1,q)\backslash\phi(S)$ and $\mathrm{PG}(n-1,q)\backslash\psi(S)$ are isomorphic; indeed, a restriction of some automorphism of $\mathrm{PG}(n-1,q)$ shows this. On the other hand, no general conclusion about whether the deletions $\mathrm{PG}(n-1,q)\backslash\phi(S)$ and $\mathrm{PG}(n-1,q)\backslash\psi(S)$ are isomorphic can be drawn if ϕ and ψ are inequivalent. This suggests the following refinement of the notion of inequivalent representations [18].

Definition 4.1. *Two representations ϕ and ψ of a geometry M, on the ground set S, in $\mathrm{PG}(n-1,q)$ are* strongly inequivalent *if the deletions $\mathrm{PG}(n-1,q)\backslash\phi(S)$ and $\mathrm{PG}(n-1,q)\backslash\psi(S)$ are not isomorphic.*

Our interest in strongly inequivalent representations lies in the following fundamental result, which follows by combining Definition 4.1 with Proposition 2.1.

Theorem 4.2. *If ϕ and ψ are strongly inequivalent representations of a geometry M, on the ground set S, in $\mathrm{PG}(n-1,q)$, then the deletions $\mathrm{PG}(n-1,q)\backslash\phi(S)$ and $\mathrm{PG}(n-1,q)\backslash\psi(S)$ are not isomorphic but have the same Tutte polynomial.*

Whittle [18] raised the following interesting open problem: find a geometric characterization of strong inequivalence. Theorem 4.3 gives a sufficient condition for representations to be strongly inequivalent.

Theorem 4.3. *Assume that ϕ and ψ are embeddings of a geometry M, on the ground set S, into $\mathrm{PG}(n-1,q)$ such that for some subset T of S, the restrictions of ϕ and ψ to the set T are inequivalent representations of $M|T$. Assume that all automorphisms of M fix each element of T. Assume, further, that M has at most $(q^{n-1}-1)/(q-1)$ points. Then ϕ and ψ are strongly inequivalent.*

Proof. Assume that ϕ and ψ are not strongly inequivalent, that is, $\mathrm{PG}(n-1,q)\backslash\phi(S)$ and $\mathrm{PG}(n-1,q)\backslash\psi(S)$ are isomorphic; we will derive a contradiction. Let N be the geometry that these deletions represent. Since M has at most $(q^{n-1}-1)/(q-1)$ points, it follows that N has at least q^{n-1} points. By Proposition 3.5, N is uniquely representable over $\mathrm{GF}(q)$, so there is an automorphism σ of $\mathrm{PG}(n-1,q)$ that maps $\mathrm{PG}(n-1,q)\backslash\phi(S)$ onto $\mathrm{PG}(n-1,q)\backslash\psi(S)$. Therefore σ maps $\phi(S)$ onto $\psi(S)$ and so gives an isomorphism of the restrictions $\mathrm{PG}(n-1,q)|\phi(S)$ and $\mathrm{PG}(n-1,q)|\psi(S)$. Since all automorphisms of M restrict to the identity on the set T, it follows that $\sigma(\phi(t))=\psi(t)$ for all $t\in T$. However, this contradicts the assumption that the restrictions of ϕ and ψ to the set T are inequivalent representations of $M|T$. □

We turn to our first application of Theorems 4.2 and 4.3.

Theorem 4.4. *Assume that q is the t-th power of a prime, that q exceeds 5, and that $q-1$ is not a Mersenne prime. Let d be the largest proper divisor of $q-1$. For each integer n with $n\geq 3$, there are at least*

$$\frac{(d-1)^{n-1}(q-1-d)(q-2-d)}{t}$$

non-isomorphic 3-connected geometries that are representable over $\mathrm{GF}(q)$, that have rank $2n+2$, that contain

$$\frac{q^{2n+2}-1}{q-1}-5n-5$$

points, and that have the same Tutte polynomial. In particular, if q is odd, then there are at least

$$\left(\frac{q-3}{2}\right)^{n}\frac{q-1}{2t}$$

such geometries.

Proof. The largest proper subgroup of the cyclic group $\mathrm{GF}^*(q)$ has order equal to the largest proper divisor, d, of $q-1$. In the case that q is odd, d is $(q-1)/2$. Thus, the number of geometries with the properties asserted in the theorem is the lower bound on the number of inequivalent representations of the n-swirl M_n over $\mathrm{GF}(q)$ given in Proposition 3.3. It suffices to extend each of the inequivalent representations of M_n to a representation of a fixed extension M_n' of M_n that has $5n+5$ elements, rank $2n+2$, and for which all automorphisms of M_n' fix M_n point-wise. By Theorem 4.3, this yields $(d-1)^{n-1}(q-1-d)(q-2-d)/t$ strongly inequivalent representations of M_n', so the desired result follows from Theorem 4.2.

We construct such a geometry M'_n through the following sequence of parallel connections with three- and four-point lines: take the parallel connection of M_n with a four-point line using basepoint e_1, and three-point lines using, in turn, $e_2, f_1, f_2, \dots, f_n$ as basepoints. Note that M'_n has $5n+5$ points and rank $2n+2$; also, $5n+5$ is less than $(q^{2n+1}-1)/(q-1)$, as needed to apply Theorem 4.3 to representations in $\mathrm{PG}(2n+1,q)$. It is easy to see that any representation of a matroid over $\mathrm{GF}(q)$ can be extended to a representation of the parallel connection of the matroid with a line that has at most $q+1$ points. Therefore it follows from Proposition 3.3 and Theorem 3.2 that M'_n has at least $(d-1)^{n-1}(q-1-d)(q-2-d)/t$ representations in $\mathrm{PG}(2n+1,q)$ for which the induced representations of M_n (in the sense of Theorem 4.3) are inequivalent.

We next check that M'_n satisfies the remaining hypothesis of Theorem 4.3: each automorphism ρ of M'_n fixes each point of M_n. Recall that the n-swirl M_n is an n-element extension of the n-whirl, where one point is added freely to each three-point line. It follows that e_1 is the only point of M'_n that is on three 4-point lines and so must be fixed by ρ. Likewise, e_2 is fixed since it is the only point that is on two 4-point lines and one 3-point line. It now follows that each of $e_3, e_4, \dots, e_n$ must be fixed by ρ since these are the only other points that are on two 4-point lines, and e_3 is on such a line with the fixed point e_2, and so on. The point f_i is fixed by ρ since it is the only point on the fixed line $\mathrm{cl}(\{e_i, e_{i+1}\})$ that is also on a 3-point line. (Subscripts in $\mathrm{cl}(\{e_i, e_{i+1}\})$ are interpreted modulo n.) Having all other points on the line $\mathrm{cl}(\{e_i, e_{i+1}\})$ fixed, it follows that g_i is also fixed.

The final issue to be addressed is that deleting the image of an embedding of M'_n in $\mathrm{PG}(2n+1,q)$ yields a geometry that is 3-connected. This follows since the number of points in such a deletion implies that it is neither a direct sum nor a 2-sum. □

Our second application of Theorems 4.2 and 4.3 uses free n-spikes.

Theorem 4.5. *Assume that q is a proper prime power other than four, say $q = p^t$. For each integer n with $n \geq 4$, there are at least*

$$\frac{(p^{t-1}-1)^{n-2}(p^t - p^{t-1})(p^t - p^{t-1} - 1)}{t}$$

non-isomorphic 3-connected geometries that are representable over $\mathrm{GF}(q)$, that have rank $n(n+3)/2$, that contain

$$\frac{q^{n(n+3)/2}-1}{q-1} - \frac{3n^2+7n+2}{2}$$

points, and that have the same Tutte polynomial.

Proof. Since the additive group of the field $\mathrm{GF}(q)$ is an abelian group of order p^t, its largest proper subgroup has order p^{t-1}. Thus, the number of geometries with the properties asserted in the theorem is the lower bound on the number of inequivalent representations of the free n-spike N_n over $\mathrm{GF}(q)$ given in Proposition 3.4. As in the proof of Theorem 4.4, it suffices to extend each of the inequivalent representations of N_n to a representation of a fixed extension N'_n of N_n that has $(3n^2+7n+2)/2$ elements, rank $n(n+3)/2$, and for which all automorphisms of N'_n fix N_n point-wise. Such an extension N'_n can be obtained by iterating the operation of parallel connection to produce i four-point lines through the point e_i, as i varies from 1 to n. The details are omitted since they are similar to those in the proof of Theorem 4.4. □

5. Variations on the basic constructions

It is easy to produce many more examples of large sets of non-isomorphic 3-connected geometries that are representable over $\mathrm{GF}(q)$ and that have the same Tutte polynomial. We mention several simple techniques.

First, rather than using all three- and four-point lines to get M'_n from M_n, or N'_n from N_n, one can use lines that contain other numbers of points. Indeed, starting with N_n, one can distinguish $e_1, e_2, \dots, e_{q-1}$ by taking parallel connections, using $e_1, e_2, \dots, e_{q-1}$ as basepoints, with lines that contain $3, 4, \dots, q+1$ points, respectively, and then distinguish the next $q-1+\binom{q-1}{2}$ points e_i by taking parallel connections, using these points as basepoints, with pairs of lines that contain between 3 and $q+1$ points, and so on. (The rank of the resulting geometry is given by a quadratic function of n, so, although this gives an improvement over the result stated in Theorem 4.5, the improvement is not dramatic.) Similarly, one can distinguish $e_1, e_2, \dots, e_n$ in N_n by iterating the operation of parallel connections where the points introduced in one parallel connection can be used as basepoints in later parallel connections.

Second, although we embedded M'_n in $\mathrm{PG}(2n+1,q)$, one could use any projective geometry of order q and rank at least $2n+2$. Similar remarks apply to N'_n.

Third, the proof of Theorem 4.4 also applies to direct sums $M'_{n_1} \oplus M'_{n_2} \oplus \cdots \oplus M'_{n_k}$ where $n_1, n_2, \dots, n_k$ are distinct. Since the number of partitions of an integer, using distinct parts, grows very rapidly, this results in another huge family of sets of matroids with the desired properties. Similar remarks apply to N'_n.

The fourth technique involves q-cones [17]. Let M be a geometry of rank n that is representable over $\mathrm{GF}(q)$ and assume that S is a subset of the ground set of $\mathrm{PG}(n,q)$ such that the restriction $\mathrm{PG}(n,q)|S$ is isomorphic to M. Let a be a point of $\mathrm{PG}(n,q)$ that is not in the hyperplane spanned by S and let T be the set

$$\bigcup_{s\in S} \mathrm{cl}_P(\{a,s\}),$$

where cl_P denotes the closure operator of $\mathrm{PG}(n,q)$. The restriction $\mathrm{PG}(n,q)|T$ is a *q-cone* of M. Thus, a q-cone of M is formed by embedding M in a hyperplane of $\mathrm{PG}(n,q)$ and restricting $\mathrm{PG}(n,q)$ to the union of the lines that are spanned by a point in the image of M and a fixed point outside the hyperplane spanned by this image. On the one hand, q-cones of non-isomorphic geometries are not isomorphic [5]; on the other hand, a geometry that has inequivalent representations over $\mathrm{GF}(q)$ may have non-isomorphic q-cones [14]. However, all q-cones of a geometry have the same Tutte polynomial. Indeed, the Tutte polynomial of a q-cone M' of a rank-n geometry M is determined by the Tutte polynomial of M; specifically, we have the following formula from [5].

$$t(M';x,y) = \frac{y(y^q-1)^n}{(y-1)^{n+1}} t(M; \frac{(x-1)(y-1)}{y^q-1}+1, y^q) + \frac{q^n(xy-x-y)}{y-1}\ t(M; \frac{x-1}{q}+1, y)$$

Assume that $\mathcal{M}_0$ is a set of k non-isomorphic geometries of rank n that are representable over $\mathrm{GF}(q)$ and that have the same Tutte polynomial (e.g., the set of geometries constructed in Theorems 4.4 or 4.5, or with the three variations mentioned above). Let the set $\mathcal{M}_1$ consist of exactly one geometry isomorphic to each q-cone of each geometry in $\mathcal{M}_0$. It follows that $\mathcal{M}_1$ is a set of k or more non-isomorphic geometries of rank $n+1$ that have the same Tutte polynomial. It is shown in [5] that the vertical connectivity of a q-cone is one greater than the vertical connectivity of the original geometry. Therefore, for any integer m, by starting from the set $\mathcal{M}_0$ and iterating the operation of q-cones, we can produce another such set $\mathcal{M}_h$ where, in addition to having k or more non-isomorphic geometries that have the same Tutte polynomial, the vertical connectivity of every geometry in $\mathcal{M}_h$ is at least m.

By combining Theorems 4.4 and 4.5 with the variations mentioned in this section, we see that for large integers n, the geometries of rank n that are representable over $\mathrm{GF}(q)$ break up into a large number of large classes with geometries in the same class having the same Tutte polynomial. (However, some geometries are distinguished from all others by their Tutte polynomials [1, 4, 15, 16].) This observation makes the following problems all the more intriguing. Let $G_{n,q}$ be a set of all isomorphism types among rank-n geometries that are representable over $\mathrm{GF}(q)$; thus, each rank-n geometry that is representable over $\mathrm{GF}(q)$ is isomorphic to precisely one geometry in $G_{n,q}$. Let the set $T_{n,q}$ be given as follows.

$$T_{n,q} = \{t(M;x,y) \mid M \in G_{n,q}\}$$

Let $g_{n,q}$ and $t_{n,q}$ be the cardinalities of $G_{n,q}$ and $T_{n,q}$, respectively. What can be said about the sequence of ratios $t_{n,q}/g_{n,q}$ for a fixed prime power q? Does this sequence decrease monotonically? Does this sequence have a limit? If this sequence has a limit, what is the limit?

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