# MATROIDS WITH NO $(q+2)$-POINT-LINE MINORS

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ABSTRACT. It is known that a geometry with rank $r$ and no minor isomorphic to the $(q+2)$-point line has at most $(q^r-1)/(q-1)$ points, with strictly fewer points if $r>3$ and $q$ is not a prime power. For $q$ not a prime power and $r>3$, we show that $q^{r-1}-1$ is an upper bound. For $q$ a prime power and $r>3$, we show that any rank-$r$ geometry with at least $q^{r-1}$ points and no $(q+2)$-point-line minor is representable over G$(q)$. We strengthen these bounds to $q^{r-1}-(q^{r-2}-1)/(q-1)-1$ and $q^{r-1}-(q^{r-2}-1)/(q-1)$ respectively when $q$ is odd. We give an application to unique representability and a new proof of Tutte's theorem: A matroid is binary if and only if the 4-point line is not a minor.

## 1. SOME EXTREMAL MATROIDS: PROJECTIVE AND AFFINE GEOMETRIES

We are concerned with matroids containing no minor isomorphic to the $(q+2)$-point line, i.e., the uniform matroid $U_{2,q+2}$. These matroids form a minor-closed class, denoted $\mathcal{U}(q)$. This class is of interest in part because of its connections with representability questions and its role in extremal matroid theory, in particular, in connection with size functions (see Section 4.2 of [6], especially Corollary 4.5).

If $q$ is a prime power, then $\mathcal{L}(q) \subseteq \mathcal{U}(q)$, where $\mathcal{L}(q)$ is the class of matroids representable over G$(q)$. Tutte [10] proved that $\mathcal{L}(2)=\mathcal{U}(2)$. The containment is strict for all other prime powers $q$. The starting point for our work is the following result [6, Theorem 4.3].

**Theorem 1.** *Rank-$r$ geometries in $\mathcal{U}(q)$ have at most $(q^r-1)/(q-1)$ points. This upper bound is attained only by projective geometries of order $q$.*

Thus for $r>3$, this bound is attained if and only if $q$ is a prime power.

To set the stage for the rest of the paper, we start by giving an alternate proof of Theorem 1, using the axioms of projective geometry.

To prove the upper bound in Theorem 1, assume $M$ is a rank-$r$ geometry in $\mathcal{U}(q)$ with $n$ points. Consider $M/x$, the contraction of $M$ by the point $x$. Since lines through $x$ contain at most $q$ points in addition to $x$, the simplification of $M/x$ has at least $(n-1)/q$ points. Thus if $M$ has more than $(q^r-1)/(q-1)$ points, the simplification of $M/x$ has more than $(q^{r-1}-1)/(q-1)$ points. Contracting $r-2$ times yields a line with at least $q+2$ points, contrary to the assumption.

The next observation follows immediately from these ideas and is useful for analyzing the case in which the upper bound is attained.

**Lemma 1.** *For any rank-$r$ geometry $M$ in $\mathcal{U}(q)$ with $(q^r-1)/(q-1)$ points, the lines through any point $x$ partition the points of $M-x$ into blocks of size $q$.*

It follows from Lemma 1 that for any rank-$r$ geometry $M$ in $\mathcal{U}(q)$ with $(q^r-1)/(q-1)$ points, each line contains exactly $q+1$ points. Furthermore, if $r=3$ each point is on exactly $q+1$ lines. From this, it is easy to check that all $(q^2+q+1)$-point planes with no $(q+2)$-point-line minor are projective planes of order $q$.

Recall that a geometry is a projective geometry if and only if any two coplanar lines intersect in a point and all lines have three or more points. Thus to prove that rank-$r$ geometries in $\mathcal{U}(q)$ with $(q^r-1)/(q-1)$ points are projective geometries, it suffices to prove that coplanar lines intersect and that each line has $q+1$ points. We have already verified the latter condition. The next lemma shows that every hyperplane of $M$

contains $(q^{r-1}-1)/(q-1)$ points. Iterating this shows that each plane is a $(q^2+q+1)$-point plane, and hence the observation in the last paragraph applies, showing that coplanar lines intersect.

**Lemma 2.** *If $M$ is a rank-$r$ geometry in $\mathcal{U}(q)$ with $(q^r-1)/(q-1)$ points, then any hyperplane $H$ of $M$ has $(q^{r-1}-1)/(q-1)$ points.*

*Proof.* We have seen that $H$ has at most $(q^{r-1}-1)/(q-1)$ points. Any point $x \in H$ partitions $M-x$ via the $(q+1)$-point lines through $x$. Note that each block is a subset of either $H$ or $M-H$. If $H$ had fewer than $(q^{r-1}-1)/(q-1)$ points, then the simplification of the contraction $M/x$ would have more than $q^{r-1}/q = q^{r-2}$ points in the complement of the hyperplane $H/x$. Iterating this yields a $(q^2+q+1)$-point plane having more than $q^2$ points off some line. However all lines have $q+1$ points. Thus $H$ has $(q^{r-1}-1)/(q-1)$ points. □

It is useful to have the affine analog of Theorem 1 given in Theorem 2.

**Theorem 2.** *Rank-$r$ geometries in $\mathcal{U}(q)$ with no $(q+1)$-point lines have at most $q^{r-1}$ points. This upper bound is attained only by affine geometries of order $q$.*

To prove the upper bound, assume $M$ is a rank-$r$ geometry in $\mathcal{U}(q)$ with $n$ points and no $(q+1)$-point lines. Since lines through any point $x$ contain at most $q-1$ points in addition to $x$, there are at least $(n-1)/(q-1)$ points in the simplification of the contraction $M/x$. By Theorem 1, $(n-1)/(q-1) \leq (q^{r-1}-1)/(q-1)$. Hence $n \leq q^{r-1}$.

This also yields the following lemma.

**Lemma 3.** *For any rank-$r$ geometry $M$ in $\mathcal{U}(q)$ with $q^{r-1}$ points and no $(q+1)$-point line, the lines through any point $x$ partition the points of $M-x$ into blocks of size $q-1$.*

**Lemma 4.** *If $M$ is a rank-$r$ geometry in $\mathcal{U}(q)$ with $q^{r-1}$ points and no $(q+1)$-point line, then any hyperplane $H$ of $M$ has $q^{r-2}$ points.*

*Proof.* For $x \in H$, the simplification of the contraction $M/x$ has $(q^{r-1}-1)/(q-1)$ points. By Lemma 2, its hyperplane $H/x$ has $(q^{r-2}-1)/(q-1)$ points. Thus $H$ has $q^{r-2}$ points. □

Recall [9] that an affine space is a collection of points, lines, and planes such that:

- (A1). Each pair of distinct points lies in a unique line.
- (A2). Each triple of distinct, non-collinear points lies in a unique plane.
- (A3). For each line and point not on the line, there is a unique line parallel to the line and containing the point.
- (A4). Any two planes in a rank-4 subspace are either parallel or intersect in a line.

To prove the second part of Theorem 2, we need to verify the third and fourth axioms for any rank-$r$ geometry $M$ in $\mathcal{U}(q)$ having $q^{r-1}$ points and no $(q+1)$-point line.

It follows from Lemma 3 and Theorem 1 that each contraction $M/x$ is a projective geometry. Therefore we get basic counting results: each point is on $q+1$ lines in a plane; each point is in $q^2+q+1$ planes in a rank-4 space, etc. By Lemma 3, each line contains exactly $q$ points.

For the third axiom, consider a line $\ell$ and a point $x$ off $\ell$. The point $x$ is on $q+1$ lines in the plane $x \vee \ell$, $q$ of which are determined by $x$ and, in turn, each of the $q$ points on $\ell$. Therefore there is a unique line through $x$ in $x \vee \ell$ not intersecting $\ell$.

To prove the fourth axiom, it suffices to show that all planes in a rank-4 flat $F$ containing the point $x$ of the plane $\pi$ meet $\pi$ in a line. By our observations above, there are $q^2+q+1$ planes in $F$ containing $x$, with $\pi$ being one. Note that $x$ is in $q+1$ lines in $\pi$. Each of these lines is in $q$ planes in $F$ other than $\pi$. This yields all $(q+1)q = q^2+q$ planes other than $\pi$ containing $x$, as needed.

## 2. Extending to the Extremal Matroids

We show that if there are between $q^{r-1}$ and $(q^r-1)/(q-1)-1$ points in a rank-$r$ geometry in $\mathcal{U}(q)$, then we can add a point to get a larger rank-$r$ geometry in $\mathcal{U}(q)$. Our primary interest is in the two corollaries that follow by iterating this and applying the second part of Theorem 1.

**Theorem 3.** *Assume $M(S)$ is a rank-$r$ geometry in $\mathcal{U}(q)$ with*

$$q^{r-1} \le |S| < \frac{q^r-1}{q-1}.$$

*Then $M(S)$ has a single-element extension to a rank-$r$ geometry $M^+(S\cup e)$ in $\mathcal{U}(q)$.*

**Corollary 1.** *Assume $q$ is a prime power and $r>3$. Any rank-$r$ geometry in $\mathcal{U}(q)$ with at least $q^{r-1}$ points is representable over* $\mathrm{G}(q)$.

**Corollary 2.** *Assume $q$ is not a prime power and $r>3$. Rank-$r$ geometries in $\mathcal{U}(q)$ have at most $q^{r-1}-1$ points.*

Theorem 4 strengthens these results for odd $q$.

**Theorem 4.** *Assume $q$ is odd and $M(S)$ is a rank-$r$ geometry in $\mathcal{U}(q)$. If*

$$q^{r-1} - \frac{q^{r-2}-1}{q-1} \le |S| < \frac{q^r-1}{q-1},$$

*then $M(S)$ has a single-element extension to a rank-$r$ geometry $M^+(S\cup e)$ in $\mathcal{U}(q)$.*

**Corollary 3.** *Assume $q$ is an odd prime power and $r>3$. Any rank-$r$ geometry in $\mathcal{U}(q)$ with at least $q^{r-1}-(q^{r-2}-1)/(q-1)$ points is representable over* $\mathrm{G}(q)$.

**Corollary 4.** *Assume $q$ is odd and not a prime power, and $r>3$. Rank-$r$ geometries in $\mathcal{U}(q)$ have at most $q^{r-1}-(q^{r-2}-1)/(q-1)-1$ points.*

The proof of Theorem 3 comes in two parts. Show that $M$ has a $q$-point line (Lemma 5). Then use this line to produce a single-element extension in $\mathcal{U}(q)$ (Lemma 6).

**Lemma 5.** *If $M(S)$ is a rank-$r\ge 3$ geometry in $\mathcal{U}(q)$ with*

$$q^{r-1} \le |S| < \frac{q^r-1}{q-1},$$

*then $M$ has a $q$-point line.*

*Remark:* For $q=2^k$, rank-3 geometries in $\mathcal{U}(q)$ having fewer than $q^2$ points need not contain a $q$-point line. Deleting the oval $\{(1,x,x^2)\,|\,x\in \mathrm{G}(2^k)\}\cup\{(0,1,0),(0,0,1)\}$ from $\mathrm{PG}(2,2^k)$ gives a geometry with $q^2-1$ points in which all lines have either $q+1$ or $q-1$ points.

*Proof.* We induct on $r$, first treating $r=3$. The basic idea is to show we may focus on the case where there is enough regularity in $M$ that we can count lines that fail to meet a fixed line that is missing at least one point; these can contain at most $q$ points since they fail to intersect the fixed line, and if all have strictly fewer points, then the fixed line must have exactly $q$ points.

Consider a point $x$ in the rank-3 geometry $M$. If $x$ is not on a $(q+1)$-point line, then either it is on a $q$-point line or the maximum number of points is $(q+1)(q-2)+1=q^2-q-1$. Thus we may assume that each point is on a $(q+1)$-point line. It follows that each point is on $q+1$ lines.

Note that any $(q+1)$-point line $\ell$ is modular, that is, $\ell\cap\ell'\neq\emptyset$ for each line $\ell'\neq\ell$. To see this, fix $y\in\ell'-\ell$. Note that $y$ and, in turn, the $q+1$ points on $\ell$ give $q+1$ lines through $y$. Thus all lines through $y$ meet $\ell$. In particular, $\ell\cap\ell'\neq\emptyset$.

We claim that there are $q^2+q+1$ lines. Since each point is on a $(q+1)$-point line, there are at least two $(q+1)$-point lines $\ell_1$ and $\ell_2$. By modularity, $\ell_1$ and $\ell_2$ intersect in a point $x$. Each line in $M$ either contains $x$ (accounting for $q+1$ lines) or is the line through a point of $\ell_1 - x$ and a point of $\ell_2 - x$ (accounting for $q^2$ lines).

Since $M$ has fewer than $q^2+q+1$ points, there is some line $\ell$ with $q+1-i$ points for some $i$ with $1 \leq i \leq q-1$. Note that each point $x \notin \ell$ is on $q+1-i$ lines intersecting $\ell$ in a point and on $i$ lines containing no point of $\ell$. Thus there are $\big(|S| - (q+1-i)\big)i \geq \big(q^2 - (q+1-i)\big)i$ pairs $(x, \ell')$ where $x$ is a point off $\ell$ and $\ell'$ is a line through $x$ containing no point of $\ell$. There are $(q+1-i)q$ lines other than $\ell$ containing a point of $\ell$ and hence $(q^2+q) - (q+1-i)q = iq$ lines containing no point of $\ell$. If each of these $iq$ lines has at most $q-1$ points, the number of pairs $(x, \ell')$ as above is at most $iq(q-1)$. Thus

$$\big(q^2 - (q+1-i)\big)i \leq iq(q-1). \tag{1}$$

This is equivalent to $i \leq 1$, forcing $i = 1$. This completes the rank-3 case.

The basic idea for the inductive step is the same as for rank 3, but the counting arguments are more involved. Assume the result holds for all ranks $r'$ with $3 \leq r' < r$, and that $M(S)$ is a rank-$r$ geometry with $q^{r-1} \leq |S| < (q^r - 1)/(q-1)$.

Consider the lines through any point $x$. Since $M \in \mathcal{U}(q)$, there are at most $(q^{r-1}-1)/(q-1)$ lines through $x$. If each of these lines had at most $q-1$ points, then the maximum number of points in $M$ would be

$$\frac{q^{r-1}-1}{q-1}(q-2) + 1,$$

which is less than $q^{r-1}$. Thus either we have the needed $q$-point line or $x$ is on some $(q+1)$-point line. Thus we may assume that each point is on some $(q+1)$-point line.

We claim that each flat $F$ of rank $r'$ with $2 \leq r' \leq r-2$ and $(q^{r'}-1)/(q-1)$ points has a cover (that is, a flat covering $F$) $F'$ with $|F'| \geq q^{r'}$. To see this, note that since $M \in \mathcal{U}(q)$, at most $(q^{r-r'}-1)/(q-1)$ flats cover $F$. If each flat covering $F$ had at most $q^{r'}-1$ points, there would be at most $q^{r'} - 1 - (q^{r'}-1)/(q-1)$ points in each cover in addition to those in $F$, and hence at most

$$\frac{q^{r-r'}-1}{q-1}\Big(q^{r'} - 1 - \frac{q^{r'}-1}{q-1}\Big) + \frac{q^{r'}-1}{q-1} \tag{2}$$

points in $M$. Since this is less than $q^{r-1}$, the required flat $F'$ exists.

If $|F'| < (q^{r'+1}-1)/(q-1)$, then, by the inductive assumption, $F'$ has a $q$-point line. Therefore we may assume that each $(q+1)$-point line of $M$ is contained in a hyperplane with $(q^{r-1}-1)/(q-1)$ points. By Theorem 1, each such hyperplane is a projective space of order $q$.

Let $H_1$ be a hyperplane with $(q^{r-1}-1)/(q-1)$ points. Starting with a $(q+1)$-point line not in $H_1$, we get a second $(q^{r-1}-1)/(q-1)$-point hyperplane $H_2$. Note that any hyperplane $H$ with $(q^{r-1}-1)/(q-1)$ points is modular since the existence of a line disjoint from $H$ would produce, upon contracting a point on the line, a minor of rank $r-1$ with at least $(q^{r-1}-1)/(q-1)+1$ points. It follows that $H_1 \cap H_2$ is a coline $C$ of $M$.

Being a hyperplane of the projective space $H_1$, $C$ has $(q^{r-2}-1)/(q-1)$ points. For any fixed $x \in H_1 - C$, the points outside $H_1$ are partitioned by the lines $x \vee z$ as $z$ ranges over the $q^{r-2}$ points in $H_2 - C$. If $C$ is in at most $q$ hyperplanes, then each line $x \vee z$ contains at most $q-1$ points other than $x$. If all contain fewer points, then $M - H_1$ contains at most $(q-2)q^{r-2}$ points. The minimum number of points in $M - H_1$ is $q^{r-1} - (q^{r-1}-1)/(q-1)$. Since

$$(q-2)q^{r-2} < q^{r-1} - \frac{q^{r-1}-1}{q-1}, \tag{3}$$

it follows that either some $x \vee z$ is a $q$-point line or the coline $C$ is covered by $q+1$ hyperplanes. We may assume the latter holds.

Indeed we may assume that each coline that is the intersection of two $(q^{r-1}-1)/(q-1)$-point hyperplanes is covered by $q+1$ hyperplanes. The same is true of all colines. For if $C$ is a coline that is not the intersection of two $(q^{r-1}-1)/(q-1)$-point hyperplanes, then since there are at least two hyperplanes of this cardinality, there is a hyperplane $H$ with $(q^{r-1}-1)/(q-1)$ points that does not contain $C$. The intervals $[C\cap H, H]$ and $[C, C\vee H]$ are isomorphic since $H$ is modular. Thus $C$ is covered by $q+1$ hyperplanes since this is true of the coline $C\cap H$ in the projective space $H$.

We claim that there are $(q^r-1)/(q-1)$ hyperplanes in $M$. Focus again on $C=H_1\cap H_2$ as above. For hyperplanes $H$ not containing $C$, we have that $H\cap C$ is a hyperplane of $C$ and $H\cap H_i$ is a hyperplane of $H_i$ $(i=1,2)$. Therefore each hyperplane not containing $C$ is determined by a hyperplane of $C$, and an extension to a hyperplane other than $C$ of $H_i$ $(i=1,2)$. Thus there are $(q^{r-3}+\cdots+q+1)\cdot q\cdot q$ hyperplanes not containing $C$. These together with the $q+1$ containing $C$ give a total of $(q^r-1)/(q-1)$ hyperplanes.

We get other counting results that are standard for projective spaces: each point is in $(q^{r-1}-1)/(q-1)$ hyperplanes, each line is in $(q^{r-2}-1)/(q-1)$ hyperplanes, and each plane is in $(q^{r-3}-1)/(q-1)$ hyperplanes. To see this, let $x$ be a point and $H$ be a hyperplane with $(q^{r-1}-1)/(q-1)$ points. If $x\notin H$, then the interval of flats over $x$ is isomorphic to the interval of flats under $H$ by the modularity of $H$. Since $H$ is a projective space and hence has $(q^{r-1}-1)/(q-1)$ hyperplanes, $x$ is in $(q^{r-1}-1)/(q-1)$ hyperplanes of $M$. If $x\in H$, then each of the other hyperplanes containing $x$ intersects $H$ is a hyperplane of $H$ containing $x$. Since $H$ is a projective space, there are $(q^{r-2}-1)/(q-1)$ hyperplanes of $H$ containing $x$. Each of these hyperplanes of $H$ is covered by $q$ hyperplanes of $M$ other than $H$. Thus $x$ is in

$$\frac{q^{r-2}-1}{q-1}q+1=\frac{q^{r-1}-1}{q-1}$$

hyperplanes of $M$. The arguments for the numbers of hyperplanes containing each line or each plane are similar.

Now consider a line $\ell$ containing $q+1-i$ points where $1\le i\le q-1$. Consider all pairs $(x,H)$ consisting of a point $x$ off $\ell$ and a hyperplane $H$ containing $x$ but no point of $\ell$. First focus on a point $x\notin\ell$. There are $(q^{r-1}-1)/(q-1)$ hyperplanes through $x$. There are $(q+1-i)(q^{r-2}-1)/(q-1)$ hyperplanes containing $x$ and at least one point of $\ell$; each of the $(q^{r-3}-1)/(q-1)$ hyperplanes containing $x$ and $\ell$ has been counted $q-i+1$ times. Hence the number of hyperplanes $H$ containing $x$ but no point of $\ell$ is

$$\frac{q^{r-1}-1}{q-1}-\left[(q+1-i)\frac{q^{r-2}-1}{q-1}-(q-i)\frac{q^{r-3}-1}{q-1}\right]=iq^{r-3}.$$

Therefore there are $(|S|-q-1+i)iq^{r-3}$ pairs $(x,H)$.

An argument as above shows that there are

$$\frac{q^{r}-1}{q-1}-\left[(q+1-i)\frac{q^{r-1}-1}{q-1}-(q-i)\frac{q^{r-2}-1}{q-1}\right]=iq^{r-2}$$

hyperplanes $H$ not meeting $\ell$. Since these hyperplanes contain no points of $\ell$, each contains fewer than $(q^{r-1}-1)/(q-1)$ points. If one of these hyperplanes has at least $q^{r-2}$ points, then it contains a $q$-point line by the inductive assumption. If each of these hyperplanes has at most $q^{r-2}-1$ points, then the number of pairs $(x,H)$ is at most $iq^{r-2}(q^{r-2}-1)$. Therefore

$$(|S|-q-1+i)iq^{r-3}\le iq^{r-2}(q^{r-2}-1). \tag{4}$$

In particular, $(q^{r-1}-q-1+i)iq^{r-3}\le iq^{r-2}(q^{r-2}-1)$, which gives $i\le 1$. Thus $\ell$ is the needed $q$-point line. □

The following lemma, which provides the second step in the proof of Theorem 3, uses Crapo's theory of single-element extensions [3,4,8]. We briefly review the terminology. Assume $M^+(S\cup e)$ is a single-element extension of $M(S)$, i.e., $M(S)$ is the restriction of $M^+(S\cup e)$ to $S$. The flats of $M^+$ are of the form $A$ or $A \cup e$ where $A$ ranges over the flats of $M$. In particular, the flats of $M$ are partitioned into three collections:

$$\mathcal{M} = \{A \,|\, A \cup e \text{ is a flat of } M^+ \text{ but } A \text{ is not a flat}\},$$
$$\mathcal{C} = \{A \,|\, A \text{ is a flat of } M^+ \text{ but } A \cup e \text{ is not a flat}\},$$
$$\mathcal{I} = \{A \,|\, \text{both } A \text{ and } A \cup e \text{ are flats of } M^+\}.$$

(For extensions of the same rank and with no additional loops, none of these collections is empty.) The collection $\mathcal{M}$ is a filter and has this property: if $A, B \in \mathcal{M}$ with $A$ and $B$ covering the flat $A \cap B$, then $A \cap B \in \mathcal{M}$. Any filter of flats with this property is called a *modular cut*. The collection $\mathcal{C}$ is called the *collar* of the extension. A flat $A$ is in the collar if and only if it is not in the modular cut but is covered by a flat in the modular cut. The collection $\mathcal{I}$ is the ideal of flats in neither the modular cut nor the collar. Thus from $\mathcal{M}$, we can find both $\mathcal{C}$ and $\mathcal{I}$. Not only does every single-element extension give rise to a modular cut, but the converse holds: any modular cut $\mathcal{M}$ of $M$ gives rise to a single-element extension of $M$, where we find $\mathcal{C}$ and $\mathcal{I}$ from $\mathcal{M}$ as above and construct the flats as specified by these three collections.

**Lemma 6.** *Assume $M(S)$ is a geometry in $\mathcal{U}(q)$ with a q-point line $\ell$, and for each coplane $X$ the rank-3 interval $[X, S]$ in the lattice of flats satisfies at least one of the following conditions:*

(a) *$[X, S]$ contains a $(q+1)$-point line, or*
(b) *$[X, S]$ contains at least $q^2 - 1$ points.*

*Then $M(S)$ has a single-element extension to a rank-r geometry $M^+(S \cup e)$ in $\mathcal{U}(q)$.*

*Proof.* Let $\ell = \{x_1, \ldots, x_q\}$ and let $\mathcal{M}$ be the collection of all flats $A$ of $M$ such that one of the following holds:

(i) $\ell \subseteq A$ or
(ii) $r(A \vee \ell) = r(A) + 1$ and $A \cap \ell = \emptyset$.

We claim that $\mathcal{M}$ is a modular cut. To see that $\mathcal{M}$ is a filter, note that (i) clearly defines a filter, while if $A$ satisfies (ii), then all $x_i \in \ell$ are in the same cover of $A$, and therefore that cover of $A$ satisfies (i) while the other covers satisfy (ii). Now assume $A, B \in \mathcal{M}$ and that $A$ and $B$ cover $A \cap B$. If $A$ and $B$ satisfy (i), the same is true of $A \cap B$. If $A$ satisfies (i) while $B$ satisfies (ii), then $r((A \cap B) \vee \ell) = r(A) = r(A \cap B) + 1$ and $(A \cap B) \cap \ell = B \cap \ell = \emptyset$, hence $A \cap B$ satisfies (ii). If $A$ and $B$ both satisfy (ii), and if we had $r((A \cap B) \vee \ell) > r(A \cap B) + 1$, then $(A \cap B) \vee \ell = A \vee \ell = B \vee \ell$ (since we have inclusion and equal ranks), and therefore $(A \cap B) \vee \ell = A \vee B$. Therefore the rank-2 interval $[A \cap B, A \vee B]$ contains the $q + 2$ flats $A, B$, and $(A \cap B) \vee x_i$ for $i = 1, \ldots, q$, contrary to $M$ having no $(q+2)$-point-line minor. Thus $\mathcal{M}$ is a modular cut. Since $\mathcal{M}$ contains no points, the extension $M^+(S \cup e)$ of $M(S)$ is a geometry.

We need to show that $M^+(S \cup e)$ is in $\mathcal{U}(q)$. By the Scum Theorem [4,8], it suffices to check that each coline of the extension $M^+(S \cup e)$ is contained in at most $q + 1$ hyperplanes. There are four types of colines in $M^+(S \cup e)$:

(1) $A$ with $A \in \mathcal{C}$,
(2) $A$ with $A \in \mathcal{I}$,
(3) $A \cup e$ with $A \in \mathcal{M}$, and
(4) $A \cup e$ with $A \in \mathcal{I}$.

The first three cases are straightforward. For $A \in \mathcal{C}$, the hyperplanes in $M^+$ above $A$ are the hyperplanes of $M$ above $A$, with the one in $\mathcal{M}$ augmented by $e$; hence there are at most $q + 1$ hyperplanes of $M^+$ above $A$. If $A \in \mathcal{I}$, then $A \cup e, A \vee x_1, \ldots, A \vee x_q$ are $q + 1$ distinct hyperplanes over $A$, and these account for all hyperplanes above $A$ in $M^+$ since any other would satisfy (ii), contrary to $A$ not having covers in $\mathcal{M}$. For a coline $A \cup e$ with $A \in \mathcal{M}$, all covers are of the form $B \cup e$ with $B \in \mathcal{M}$, and therefore since the coline $A$ of $M$ has at most $q + 1$ hyperplanes $B$ covering it, the same is true of the coline $A \cup e$ of $M^+$.

Finally consider the fourth case, a coline $A \cup e$ with $A \in \mathcal{I}$. Thus $A$ is a coplane of $M$. Consider the rank-3 interval $[A, S]$ of flats in $M$. The points are the covers of $A$. Note that $A \vee x_1, \ldots, A \vee x_q$ are collinear covers of $A$, being on the line $A \vee \ell$. Furthermore no other cover $C$ of $A$ is on $A \vee \ell$, for then $C \vee \ell$ covers $C$ and $C \cap \ell = \emptyset$, forcing $C$ in the modular cut, contrary to $A$ being in the ideal.

We claim that the interval $[A, S \cup e]$ of flats in $M^+$ is the extension of $[A, S]$ via the $q$-point line $A \vee \ell$. To see this, let $F$ be any flat of $M$ in $[A, S]$. Note that $\ell \subseteq F$ if and only if $A \vee \ell \subseteq F$. Since $F \vee \ell = F \vee (A \vee \ell)$, we have $r(F \vee \ell) = r(F) + 1$ if and only if $r(F \vee (A \vee \ell)) = r(F) + 1$. Finally $F \cap \ell = \emptyset$ if and only if $F \cap (A \vee \ell) = A$ since $A \vee \ell = (A \vee x_1) \cup \cdots \cup (A \vee x_q)$. Thus a flat in $[A, S]$ is in the modular cut defined by $\ell$ if and only if it is in that defined by $A \vee \ell$.

It follows that it suffices to show that $e$ is on at most $q + 1$ lines in the geometry $M^+(S \cup e)$ when $M$ has rank 3, a $q$-point line $\ell = \{x_1, \ldots, x_q\}$, and either a $(q+1)$-point line or at least $q^2 - 1$ points.

Define a relation on $S - \ell$ by $y \sim y'$ if and only if $y \vee^+ e = y' \vee^+ e$. (We denote joins in $M^+$ by $\vee^+$.) Thus points of $S - \ell$ are related if and only if they give rise to the same line containing $e$. To show that $e$ is in at most $q + 1$ lines, it suffices to show that the equivalence relation $\sim$ has at most $q$ equivalence classes.

We claim that if $y \not\sim y'$, then $y \vee y'$ contains some $x_i$. Assume $y \vee y'$ contains no $x_i$. Therefore $(y \vee y') \cap \ell = \emptyset$. Since $y \vee y'$ is covered by $(y \vee y') \vee \ell$, $y \vee y'$ is in the modular cut $\mathcal{M}$. Therefore $y \vee^+ y' \vee^+ e$ is a line. Thus $y \vee^+ e = y \vee^+ y' = y' \vee^+ e$, proving $y \sim y'$.

Note that if $y \sim y'$ and $y \neq y'$, then $y \vee y'$ contains no $x_i$. Thus the nontrivial equivalence classes are the lines not meeting $\ell$.

We now show that $\sim$ has at most $q$ equivalence classes. First assume $M$ has a $(q+1)$-point line $\ell'$. Note that $\ell'$ intersects $\ell$ at some point $x_i$. If there were more than $q$ equivalence classes, there is a point $y \in M - \ell$ whose class contains none of the $q + 1$ points on $\ell'$. The lines spanned by $y$ and, in turn, the $q + 1$ points of $\ell'$ yield $q + 1$ distinct points on $\ell$, contrary to this being a $q$-point line. Thus there are at most $q$ classes, as needed.

Assume $M$ has no $(q+1)$-point lines. Thus $|S| \geq q^2 - 1$ since (b) holds, while $|S| \leq q^2$ by Theorem 2. If $|S| = q^2$, then by Theorem 2, $M$ is an affine plane. Therefore the equivalence classes under $\sim$ are precisely the $q - 1$ lines parallel to $\ell$.

Finally assume $|S| = q^2 - 1$. Since there are no $(q+1)$-point lines, each point is on one line containing $q - 1$ points and $q$ lines containing $q$ points. In particular, each point is on $q + 1$ lines, and so none of the equivalence classes under the relation $\sim$ is trivial. Furthermore since each of the $q^2 - 1$ points is on one $(q-1)$-point line, the number of $(q-1)$-point lines is $(q^2 - 1)/(q - 1) = q + 1$. Likewise the number of $q$-point lines is $q^2 - 1$. Hence $M$ has $q^2 + q$ lines. Since $q^2 + 1$ are accounted for by $\ell$ and the lines intersecting it, there are exactly $q - 1$ equivalence classes under $\sim$. □

*Proof of Theorem 3.* By Lemma 5, $M(S)$ has a $q$-point line. Since $M$ has at least $q^{r-1}$ points, each rank-3 interval $[X, S]$ in the lattice of flats has at least $q^2$ points. Therefore Lemma 6 applies, giving the needed extension. □

The proof of Theorem 4 follows the same general outline as that of Theorem 3; we focus on how the two differ.

**Lemma 7.** *Assume $q$ is odd and $M(S)$ is a rank-$r$ geometry in $\mathcal{U}(q)$ with*

$$q^{r-1} - \frac{q^{r-2} - 1}{q - 1} \leq |S| < \frac{q^r - 1}{q - 1}.$$

*Then $M$ has a $q$-point line.*

*Proof.* We induct on $r$, first treating $r = 3$. By Lemma 5, we may assume the rank-3 geometry $M$ has exactly $q^2 - 1$ points. As in the proof of Lemma 5, we may assume that each point is on a $(q+1)$-point line. It follows that each point is on $q + 1$ lines, and there are exactly $q^2 + q + 1$ lines. Let $\ell$ be a line with

$q+1-i$ points where $1 \le i \le q-1$. By considering pairs $(x, \ell')$ where $\ell' \cap \ell = \emptyset$ and $x \in \ell'$, we get $\big((q^2-1)-(q+1-i)\big)i \le iq(q-1)$ in place of (1). This is equivalent to $i \le 2$.

Thus we may assume that each line has either $q+1$ or $q-1$ points. If $k$ lines through $x$ have $q+1$ points, then there are $kq+(q+1-k)(q-2)+1$ points. Setting this to $q^2-1$ gives $2k=q$, contrary to $q$ being odd. This completes the rank-3 case.

Assume the result holds for all ranks $r'$ with $3 \le r' < r$, and that $M$ is a rank-$r$ geometry with $q^{r-1}-(q^{r-2}-1)/(q-1) \le |S| < (q^r-1)/(q-1)$. Since

$$\frac{q^{r-1}-1}{q-1}(q-2)+1 < q^{r-1}-\frac{q^{r-2}-1}{q-1},$$

it follows that each point is on either a $(q+1)$-point line or a $q$-point line. We may assume that each point is on a $(q+1)$-point line.

From the inequality

$$\frac{q^{r-r'}-1}{q-1}\Big(q^{r'}-\frac{q^{r'-1}-1}{q-1}-1-\frac{q^{r'}-1}{q-1}\Big)+\frac{q^{r'}-1}{q-1} < q^{r-1}-\frac{q^{r-2}-1}{q-1}$$

in place of (2), it follows that each flat $F$ of rank $r'$ with $2 \le r' \le r-2$ and $(q^{r'}-1)/(q-1)$ points has a cover $F'$ with at least $q^{r'}-(q^{r'-1}-1)/(q-1)$ points. If $|F'| < (q^{r'+1}-1)/(q-1)$, then $F'$ has a $q$-point line by the inductive assumption. Therefore we may assume that each $(q+1)$-point line of $M$ is contained in a hyperplane with $(q^{r-1}-1)/(q-1)$ points. Such a hyperplane is a projective space of order $q$.

As in the proof of Lemma 5, there are two hyperplanes with $(q^{r-1}-1)/(q-1)$ points, say $H_1$ and $H_2$. These cover a coline $C = H_1 \cap H_2$ having $(q^{r-2}-1)/(q-1)$ points. The minimum number of points in $M-H_1$ is $q^{r-1}-(q^{r-2}-1)/(q-1)-(q^{r-1}-1)/(q-1)$. Using

$$(q-2)q^{r-2} < q^{r-1}-\frac{q^{r-2}-1}{q-1}-\frac{q^{r-1}-1}{q-1}$$

in place of (3), it follows that either there is a $q$-point line outside $H_1$ or $C$ is covered by $q+1$ hyperplanes. We may assume the latter holds. Further we may assume that each coline that is the intersection of two $(q^{r-1}-1)/(q-1)$-point hyperplanes is covered by $q+1$ hyperplanes.

Now consider a line $\ell$ with $q+1-i$ points where $1 \le i \le q-1$. Consider all pairs $(x, H)$ consisting of a point $x$ and a hyperplane $H$ with $H \cap \ell = \emptyset$ and $x \in H$. In place of (4), we get

$$(|S|-q-1+i)iq^{r-3} \le iq^{r-2}\Big(q^{r-2}-\frac{q^{r-3}-1}{q-1}-1\Big).$$

Now

$$\Big(q^{r-1}-\frac{q^{r-2}-1}{q-1}-q-1+i\Big)iq^{r-3} \le iq^{r-2}\Big(q^{r-2}-\frac{q^{r-3}-1}{q-1}-1\Big)$$

is equivalent to $i \le 2$. In particular, if $|S| > q^{r-1}-(q^{r-2}-1)/(q-1)$, then $i=1$. Thus we may assume that $|S| = q^{r-1}-(q^{r-2}-1)/(q-1)$ and each line with fewer than $q+1$ points has $q-1$ points.

Note that a point $x \in M$ outside a $(q^{r-1}-1)/(q-1)$-point hyperplane is on exactly $(q^{r-1}-1)/(q-1)$ lines. If such a point $x$ is on $k$ lines having $q-1$ points, then there are

$$k(q-2)+\Big(\frac{q^{r-1}-1}{q-1}-k\Big)q+1$$

points. Setting this to $q^{r-1}-(q^{r-2}-1)/(q-1)$ gives $2k = q^{r-2}+2q^{r-3}+2q^{r-4}+\cdots+2q+2$, contrary to $q$ being odd. Thus there is a $q$-point line. □

*Proof of Theorem 4.* By Lemma 7, $M(S)$ has a $q$-point line. Since $M$ has at least $q^{r-1}-(q^{r-2}-1)/(q-1)$ points, each contraction has at least

$$\frac{q^{r-1}-(q^{r-2}-1)/(q-1)-1}{q} = q^{r-2} - \frac{q^{r-3}-1}{q-1}$$

points. Thus each rank-3 interval $[X, S]$ has at least $q^2-1$ points. Therefore Lemma 6 applies, giving the needed extension. □

## 3. A Theorem of Tutte

For $q = 2$, condition (b) of Lemma 6 is fulfilled since $q^2 - 1 = 3$. This proves Lemma 8.

**Lemma 8.** *Any rank-$r$ geometry $M(S) \in \mathcal{U}(2)$ with a 2-point line has a single-element extension to a rank-$r$ geometry $M^+(S \cup e) \in \mathcal{U}(2)$.*

Using this, we prove the following theorem of Tutte [10].

**Theorem 5.** *A matroid is binary if and only if the 4-point line is not a minor.*

*Proof.* It is immediate that the 4-point line is not binary and that we may focus on geometries. Assume $M$ is a rank-$r$ geometry with no 4-point-line minor. If $M$ has the maximal number of points, namely $2^r-1$, then by Theorem 1 $M$ is PG$(r-1,2)$. If all lines contain 3 points, then $M$ has $2^r-1$ points. Therefore if $M$ has fewer than $2^r-1$ points, then $M$ has at least one 2-point line. Lemma 8 allows us to extend $M$ without creating a 4-point-line minor. Iterating this as needed, we can extend $M$ to PG$(r-1,2)$. □

## 4. Uniquely Representable Combinatorial Geometries

A rank-$r$ geometry $M$ representable over the field $F$ is uniquely $F$-representable [2,8] if for any two embeddings $\phi, \psi$ of $M$ in PG$(r-1,F)$, there is an automorphism $\sigma$ of PG$(r-1,F)$ with $\phi = \sigma\psi$. Binary geometries are uniquely representable over all fields over which they are representable; ternary geometries are uniquely G$(3)$-representable [2]. A geometry representable over G$(4)$ is uniquely G$(4)$-representable if and only if it is neither a direct sum nor a 2-sum of non-binary matroids [5]. As noted in [8], a rank-$r$ geometry with at least $(4^{r-1}+14)/3$ points that is representable over G$(4)$ is uniquely G$(4)$-representable. Theorem 6 is a general result of this type.

**Theorem 6.** *A rank-$r > 3$ geometry $M$ in $\mathcal{U}(q)$ with at least $q^{r-1}$ points is uniquely* G$(q)$*-representable. If $q$ is odd, a rank-$r > 3$ geometry $M$ in $\mathcal{U}(q)$ with at least $q^{r-1}-(q^{r-2}-1)/(q-1)$ points is uniquely* G$(q)$*-representable.*

*Proof.* By Corollary 1 (Corollary 3 for odd $q$), $M$ is G$(q)$-representable. Let $\phi$ and $\psi$ be embeddings of $M$ in PG$(r-1,q)$. By Lemma 5 (Lemma 7 for odd $q$), $M$ contains a $q$-point line, say $\ell$. Extend $M(S)$ to $M^+(S \cup e)$ as in the proof of Lemma 6. Note that $\phi$ and $\psi$ extend to embeddings of $M^+$ in PG$(r-1,q)$ by mapping $e$ to the points $\overline{\phi(\ell)} - \phi(\ell)$ and $\overline{\psi(\ell)} - \psi(\ell)$ respectively. By iterating this, we obtain embeddings $\phi'$ and $\psi'$ of $M^+(S \cup E) \simeq$ PG$(r-1,q)$ into PG$(r-1,q)$. Thus $\phi'\psi'^{-1}$ is an automorphism of PG$(r-1,q)$, and $\phi = (\phi'\psi'^{-1})\psi$, as needed. □

We can improve the bound in the case of odd $q$ if we know the matroid is G$(q)$-representable. This result, the next theorem, follows from Lemma 9 below and the ideas in the proof of Theorem 6. (Since all ternary geometries are uniquely G$(3)$-representable, it suffices to focus on $q > 3$. Indeed, $q > 3$ is crucial in one of the inequalities used in the proof of Lemma 9.)

**Theorem 7.** *If $q$ is odd, a rank-$r \geq 3$ geometry $M$ in $\mathcal{L}(q)$ with more than*

$$q^{r-1} - \frac{q^{r-3}\sqrt{2q}}{4} - \frac{q^{r-2}-1}{q-1}$$

*points is uniquely* $\mathrm{G}(q)$*-representable.*

**Lemma 9.** *Assume $q>3$ is odd and $M(S)$ is a rank-$r$ geometry in $\mathcal{L}(q)$ with*

$$q^{r-1}-\frac{q^{r-3}\sqrt{2q}}{4}-\frac{q^{r-2}-1}{q-1}<|S|<\frac{q^r-1}{q-1}.$$

*Then $M$ has a $q$-point line.*

*Proof.* The basic idea follows that of the proof of Lemma 5; we focus on the differences. We induct on $r$. The case $r=3$ is the main result of [1].

Assume the result holds for all ranks $r'$ with $3\le r'<r$, and that $M$ is a rank-$r$ geometry with $q^{r-1}-q^{r-3}\sqrt{2q}/4-(q^{r-2}-1)/(q-1)<|S|<(q^r-1)/(q-1)$. Since

$$\frac{q^{r-1}-1}{q-1}(q-2)+1<q^{r-1}-\frac{q^{r-3}\sqrt{2q}}{4}-\frac{q^{r-2}-1}{q-1},$$

it follows that each point is on either a $(q+1)$-point line or a $q$-point line. We may assume that each point is on a $(q+1)$-point line.

From the inequality

$$\frac{q^{r-r'}-1}{q-1}\Big(q^{r'}-\frac{q^{r'-2}\sqrt{2q}}{4}-\frac{q^{r'-1}-1}{q-1}-1-\frac{q^{r'}-1}{q-1}\Big)+\frac{q^{r'}-1}{q-1}<q^{r-1}-\frac{q^{r-3}\sqrt{2q}}{4}-\frac{q^{r-2}-1}{q-1}$$

in place of (2), it follows that each flat $F$ of rank $r'$ with $2\le r'\le r-2$ and $(q^{r'}-1)/(q-1)$ points has a cover $F'$ with at least $q^{r'}-q^{r'-2}\sqrt{2q}/4-(q^{r'-1}-1)/(q-1)$ points. If $|F'|<(q^{r'+1}-1)/(q-1)$, then $F'$ has a $q$-point line. Therefore we may assume that each $(q+1)$-point line of $M$ is contained in a hyperplane with $(q^{r-1}-1)/(q-1)$ points.

As in the proof of Lemma 5, there are two hyperplanes with $(q^{r-1}-1)/(q-1)$ points, say $H_1$ and $H_2$. These cover a coline $C=H_1\cap H_2$ having $(q^{r-2}-1)/(q-1)$ points. The minimum number of points in $M-H_1$ is $q^{r-1}-q^{r-3}\sqrt{2q}/4-(q^{r-2}-1)/(q-1)-(q^{r-1}-1)/(q-1)$. Using

$$(q-2)q^{r-2}<q^{r-1}-\frac{q^{r-3}\sqrt{2q}}{4}-\frac{q^{r-2}-1}{q-1}-\frac{q^{r-1}-1}{q-1}$$

in place of (3), it follows that either there is a $q$-point line outside $H_1$ or $C$ is covered by $q+1$ hyperplanes. We may assume the latter holds. Further we may assume that each coline that is the intersection of two $(q^{r-1}-1)/(q-1)$-point hyperplanes is covered by $q+1$ hyperplanes. It follows that there are $(q^r-1)/(q-1)$ hyperplanes in $M$.

Now consider a line $\ell$ with $q+1-i$ points where $1\le i\le q-1$. Consider all pairs $(x,H)$ consisting of a point $x$ and a hyperplane $H$ with $H\cap\ell=\emptyset$ and $x\in H$. In place of (4), we get

$$(|S|-q-1+i)iq^{r-3}\le iq^{r-2}\Big(q^{r-2}-\frac{q^{r-4}\sqrt{2q}}{4}-\frac{q^{r-3}-1}{q-1}-1\Big).$$

Since $|S|>q^{r-1}-q^{r-3}\sqrt{2q}/4-(q^{r-2}-1)/(q-1)$, we get

$$(q^{r-1}-\frac{q^{r-3}\sqrt{2q}}{4}-\frac{q^{r-2}-1}{q-1}-q-1+i)iq^{r-3}<iq^{r-2}(q^{r-2}-\frac{q^{r-4}\sqrt{2q}}{4}-\frac{q^{r-3}-1}{q-1}-1).$$

This is equivalent to $i<2$. Thus $i=1$, completing the induction. □

With each rank $r$ and prime power $q$, there is a smallest number $u_{r,q}$ such that all rank-$r$ geometries that are representable over $\mathrm{G}(q)$ and have at least $u_{r,q}$ points are uniquely $\mathrm{G}(q)$-representable. Thus $u_{r,2}=u_{r,3}=r$, and $u_{r,4}=(4^{r-1}+14)/3$. The numbers $u_{r,q}$ are of interest in matroid reconstruction problems (see [7]).

Consider $M_q$, the parallel connection of $\mathrm{PG}(r-2,q)$ with $\mathrm{PG}(1,q)$. By comparing the number of embeddings of $M_q$ in $\mathrm{PG}(r-1,q)$ with the number of automorphisms of $\mathrm{PG}(r-1,q)$, we see that $M_q$ is not uniquely G($q$)-representable if $q > 3$. Therefore $u_{r,q} \geq |M_q| + 1 = (q^{r-1} + q^2 - 2)/(q-1)$. Putting this together with Theorem 7, we get:

$$\frac{q^{r-1}+q^2-2}{q-1} \leq u_{r,q} \leq q^{r-1} \qquad \text{for even } q > 4$$

$$\frac{q^{r-1}+q^2-2}{q-1} \leq u_{r,q} < q^{r-1} - \frac{q^{r-3}\sqrt{2q}}{4} - \frac{q^{r-2}-1}{q-1} \qquad \text{for odd } q > 3.$$

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