Spectral Multiplicity for Non-abelian Morse Sequences

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INTRODUCTION.

The problem of determining the spectral multiplicity \( M \) for the dynamical system \( T \) generated by a generalized Morse sequence \( x \) is discussed in [J], [L1], [G1], [G2] and [KwSi]. All of these papers consider Morse sequences based on finite cyclic groups. For such sequences the only values of the multiplicity which have been observed are \( M = 1 \) and \( M = 2 \). The question of whether there are any other possibilities is apparently still open, but the answer may be negative. For sequences based on a single constant block over \( \mathbb{Z}/p \), \( p \) prime and \( p \geq 3 \), Kwiatkowski and Sikorski [KwSi] have proved that these are indeed the only possibilities.

The purpose of this paper is to show that if we expand the class of generalized Morse sequences to include sequences based on finite non-abelian groups, then arbitrary finite spectral multiplicity can be achieved. This kind of generalized Morse sequence is a special case of sequences first studied in [M]. We call them non-abelian Morse sequences. Our motivation for using non-abelian Morse sequences comes from the general principle, [R3], that non-abelian group extensions have nonsimple spectrum. This shows in particular that non-abelian Morse sequences have spectral multiplicities \( M \) satisfying lower bounds which depend only on the group. The actual construction given here is an adaptation of the construction from [R1].

1. NON-ABELIAN MORSE SEQUENCES.

Let \( G \) be an arbitrary finite group of order \( r \). For convenience we will identify \( G \) with the set \( \{1, \ldots, r\} \), where 1 will be identified with the identity in \( G \). Let \( A = (a_1, \ldots, a_l) \) and \( B = (b_1, \ldots, b_m) \), \( a_i, b_j \in \{1, \ldots, r\} \). We call \( A \) and \( B \) respectively blocks of length \( l \) and \( m \). We define block multiplication by

\[
A \times B = (A \cdot b_1) \circ (A \cdot b_2) \circ \cdots \circ (A \cdot b_m),
\]

where \( A \cdot b_j = (a_1 \cdot b_j, \ldots, a_l \cdot b_j) \), with \( a \cdot b \) denoting the product in \( G \), and where "\( \circ \)" denotes concatenation. For \( n \geq 0 \) let \( A_n = (a_1^n, \ldots, a_l^n) \) where \( a_1^n = 1 \) and \( s_n > 1 \). By induction we define \( B_1 = A_1 \) and \( B_{n+1} = B_n \times A_n \). Note that since \( A_n \) begins with 1, the block \( B_{n+1} \) begins with \( B_n \) so we can define a right infinite sequence \( x \) which we denote by

\[
x = A_1 \times A_2 \times A_3 \times A_4 \times \ldots
\]

We assume conditions (1) and (2) from [M] which guarantee that \( x \) is aperiodic and almost periodic. In particular we assume the following

\[
(1) \quad \forall g \in G \exists n > 1 \text{ and } j \text{ with } a_j^n = g.
\]

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The sequence $x$ has a two sided almost periodic extension (cf. [M]) which we also denote by $x \in G^\mathbb{Z}$. Let $X$ denote the orbit closure of $x$ in $G^\mathbb{Z}$ under the shift. The shift restricted to $X$ will be denoted by $T$. Since $x$ is almost periodic it follows from a theorem of Gottschalk [G] that $T$ is a minimal homeomorphism of $X$. We call $T$ the transformation generated by $x$, and we will often refer to dynamical properties of $T$ as properties of the Morse sequence $x$.

There is a natural $T$ invariant measure $\mu$ on $X$ defined as follows: Let $E(j, \hat{x})$ denote the intersection of $X$ with the cylinder set of all $z \in G^\mathbb{Z}$ so that the string $\hat{x}$ begins at $j$ in $z$. Then $\mu(E(j, \hat{x}))$ is equal to the frequency of $\hat{x}$ in $x$. In general $\mu$ may or may not be ergodic for $T$. (We will construct specific ergodic examples in the second section.) In the ergodic case we have the following:

**Lemma 1.** If $\mu$ is ergodic for $T$ then $T$ is uniquely ergodic.

One way to prove this is through the theory of group extensions. Suppose a transformation $T$ on $(X, \mu)$ has a factor $T_0$ on $(X_0, \mu_0)$ where $(X, \mu) = (X_0 \times G, \mu_0 \times \gamma)$ and $\gamma$ denotes Haar measure. Suppose also that $T(x, g) = (T_0 x, \phi(x) \cdot y)$ for some measurable $\phi : X_0 \to G$, (i.e. $T$ acts by $G$ multiplication in the $G$ direction). We say $T$ is a group extension or $G$ extension of $T_0$.

**Lemma 2.** The transformation $T$ on $(X, \mu)$ generated by the generalized Morse sequence $x$ is a $G$ extension of a transformation $T_0$ on $(X_0, \mu_0)$. The transformation $T_0$ is a uniquely ergodic rotation on the compact group $X_0$ of $\{s_n\}$-adic integers with Haar measure $\mu_0$, (i.e. $T_0$ is an $\{s_n\}$-adic adding machine).

**Proof:** For the first statement, it is enough to show that $T$ commutes with a free $G$ action (cf. [R4]), and it follows from (1) and the almost periodicity of $x$ that $(L_g x)_j = x_j \cdot g$ defines such an action. The second statement follows easily from the first by standard arguments. ■

**Proof of Lemma 1:** We use the following fact about group extensions (cf. eg. [R5]): If $T$ is ergodic for $\mu = \mu_0 \times \gamma$ and if $\mu_1$ is an arbitrary $T$ invariant Borel probability measure which projects to $\mu_0$, then $\mu_1 = \mu$. Now since every $T$ invariant measure projects to a $T_0$ invariant measure, the Lemma follows from the unique ergodicity of $T_0$. ■

For a transformation $T$, let $M$ denote the maximal spectral multiplicity of the corresponding induced unitary operator on $L_2$, (cf. [R1]). When $T$ is generated by a generalized Morse sequence $x$, we refer to $M$ as the spectral multiplicity of $x$, saying $x$ has simple spectrum if $M = 1$ and nonsimple spectrum otherwise. As we noted in the introduction, we are mostly interested in the case where the sequence $x$ is based on a non-abelian group $G$. We call such an $x$ a non-abelian Morse sequence.

Let $D_G$ denote the maximum of the dimensions of the irreducible representations of $G$. Since $G$ is nonabelian, $D_G > 1$. If a transformation $T$ is a $G$ extension of any transformation $T_0$, it follows from [R4] that the the spectral multiplicity $M$ of $T$ satisfies $M \geq D_G$.

**Corollary 3.** If $x$ is a non-abelian Morse sequence based on $G$ then the spectral multiplicity of $x$ satisfies $M \geq D_G > 1$. In particular every non-abelian Morse sequence has
non-simple spectrum.

2. CUTTING AND STACKING.

There is a well known correspondence between Morse sequences and certain kinds of cutting and stacking constructions.

**Definition 4.** The transformation $T$ on $(X, \mu)$ has uniform rank $r$ if $r$ is the least positive integer so that there exists a sequence $\{E^g_n, \ldots, E^g_r\}$ of collections of $r$ disjoint subsets of $(X, \mu)$, and a sequence of positive integers $q_n \to \infty$ so that

(i) $q_n \mu(E^g_n) \to 1/r$ for each $g \in \{1, \ldots, r\}$,

(ii) the sets $T^j E^g_n$, $0 \leq j < q_n$ and $1 \leq g \leq r$ are disjoint,

and

(iii) the partition $\xi_n$ of $(X, \mu)$ into the sets in (ii) satisfies $\xi_n \to \epsilon$ (cf. [R.1]).

We denote the uniform rank of $T$ by $R$. Our interest in rank arises from the well known inequality $M \leq R$ (cf. [C]) relating rank and spectral multiplicity.

Let $r$ be a positive integer. The following construction produces a class of transformations $T$ with rank $R \leq r$. For each $n \geq 0$, $B^1_n, \ldots, B^r_n$ will denote blocks of symbols from $\{1, \ldots, r\}$ of length $q_n$ to be constructed by induction. We start with $B^q_0 = (g)$, $1 \leq g \leq r$. Let $s_n > 1$ and define $q_{n+1} = s_n q_n$. Let

$$\Sigma_n = \{\sigma^n_2, \ldots, \sigma^n_{s_n}\}$$

be permutations of $\{1, \ldots, r\}$ and define the $r \ (n+1)$-blocks:

$$B^{g}_{n+1} = B^g_n \circ B^{\sigma^n_2(g)}_n \circ \ldots \circ B^{\sigma^n_{s_n}(g)}_n. \tag{2}$$

By the standard cutting and stacking construction we use the blocks (2) defined above to construct a Lebesgue measure $\mu$ preserving transformation $T$ of the unit interval $X$ with $R \leq r$. In particular, for this construction $E^{g}_n$ (Definition 4) is the set of points $z \in X$ which have $B^g_n$ for a $\{1, \ldots, r\}$-name of length $q_n$.

We will assume that

$$s_n > 2^n. \tag{3}$$

Let $\Gamma$ be a subgroup of the symmetric group $S_r$ which acts transitively on $\{1, \ldots, r\}$. We say $\Sigma_n \subseteq \Gamma$ if each $\sigma \in \Sigma_n$ comes from $\Gamma$. Let $N_1$ be an arbitrary infinite subset of the positive integers $\mathbb{N}$. By [R2], there exists a choice of $\Sigma_n \subseteq \Gamma$ for each $n \in N_1$ so that $T$ is ergodic for any choice of $\Sigma_n$ for $n \not\in N_1$. Heuristically, for the ergodicity of $T$ it is only necessary to insure that infinitely often there is sufficient “mixing” between all the “towers”.

**Comment.** Let $T_0$ be the factor of $T$ corresponding to the identification of the $r$ $n$-towers into a single $n$-tower at each construction step. Then $T_0$ is the $\{s_n\}$-adic adding machine. If $N_2$ is an infinite subset of $\mathbb{N}$ disjoint from $N_1$, then one can show that there exists a
choice of $\Sigma_n \subseteq \Gamma$ for $n \in N_2$ so that $T_0$ is the maximal discrete spectrum factor of $T$. In this case we call $T$ a continuous extension of $T_0$.

Now as in §1 we assume that the set $\{1, \ldots, r\}$ has been identified with a finite group $G$ of order $r$. We use the blocks $A_n = (a^n_1, \ldots, a^n_{s_n})$ from the Morse sequence construction to define the permutations $\sigma^n_j(g) = a^n_j \cdot g$ for cutting and stacking. In other words the permutations in $\Sigma_n$ act by multiplication in $G$, and $\Gamma = G$. Clearly $T$ is a $G$ extension of $T_0$ and it is easy to see by comparing the the two constructions that $B^1_n = B_n$ and $B^g_n = B_n \cdot g$ for all $n$ and $1 \leq g \leq r$.

**Lemma 5.** The cutting and stacking transformation $T$ is isomorphic to the corresponding transformation $\mathcal{T}$ generated by the Morse sequence $x$.

For this we just note that points in both constructions have the same names. Thus we have shown how to use cutting and stacking to construct ergodic Morse sequences which are continuous extensions.

**Corollary 6.** There exists a choice of blocks $A_n$ making the Morse sequence $x$ minimal and uniquely ergodic. Every such sequence satisfies $1 < D_G \leq M \leq R \leq r$.

### 3. Exact Estimates of the Multiplicity.

We first recall the following construction from [R1] (cf. also [R4]) which produces groups $G$ with $D_G = m$ for arbitrary $m > 1$: Given $m$, there exists a prime number $p$ so that $p$ divides $m$, (cf. [R1]). Let $\alpha \in \mathbb{Z}/p$ be a generator for the subgroup isomorphic to $\mathbb{Z}/m$ of the group of units of $\mathbb{Z}/p$. Recall that the entire group of units $(\mathbb{Z}/p)^\times$ of $\mathbb{Z}/p$ is isomorphic to $\mathbb{Z}/(p - 1)$. Let $(\mathbb{Z}/m) \times_\alpha (\mathbb{Z}/p)$ denote the semi-direct product of $\mathbb{Z}/p$ and $\mathbb{Z}/m$. This group has elements $(y, z)$ where $y \in \mathbb{Z}/m$ and $z \in \mathbb{Z}/p$, and the product is defined by

\[(y_1, z_1) \cdot (y_2, z_2) = (y_1 + y_2, \alpha^{y_2}z_1 + z_2).\]

The following lemma is proved in [R4].

**Lemma 7.** $D_G = m$ for $G = (\mathbb{Z}/m) \times_\alpha (\mathbb{Z}/p)$.

**Lemma 8.** Assume that (3) holds. Let $G = (\mathbb{Z}/m) \times_\alpha (\mathbb{Z}/p)$, let $N_3$ be an infinite subset of $\mathbb{N}$ disjoint from $N_1$ and $N_2$, and let $T$ be constructed as above. Then there exists a choice of $\Sigma_n$ for $n \in N_3$ so that the rank $R$ of $T$ satisfies $R \leq m$.

**Proof:** We will identify the element $(u, t) \in G$ with the number $up + t + 1 \in \{1, \ldots, r\}$. Note that $(y, z) \cdot (0, t) = (y, z + t)$. In particular, the elements $1, \theta, \ldots, \theta^{p-1}$, where $\theta = (0, 1)$, are identified with $1, 2, \ldots, p$. For $n$ in $N_1$ let

\[(5) \quad A_n = (1, \ldots, p, 1, \ldots, p, 1 \ldots),\]

i.e. $a^t_n - 1 = (i - 1) \text{ mod } p$. Recalling that $B^g_n = B_n \cdot g$, we have

\[
B^1_{n+1} = [(B_n \cdot 1) \circ (B_n \cdot 2) \circ \cdots \circ (B_n \cdot p)]^{k_n+2} \circ (B_n \cdot 1) \circ \cdots \circ (B_n \cdot l_n)
= (B_n \cdot 1) \circ \cdots \circ (B_n \cdot (p - 1)) \circ \circ [\circ (B_n \cdot (p - 1))^{k_n+1} \circ (B_n \cdot 1) \circ \cdots \circ (B_n \cdot l_n)]
\]
where \( l_n < p \) and \( p(k_n + 2) + l_n = s_n \). Then (3) implies \( k_n \to \infty \) as \( n \to \infty \), \( n \in N_3 \). More generally it follows from (4) that for \( 0 \leq u \leq m - 1 \) and \( 1 \leq t \leq p \),

\[
B_{n+1}^{up+t} = (B_n \cdot (up+t)) \circ \cdots \circ (B_n \cdot (up+t)) \circ \cdots \circ (B_n \cdot (up+p-1)) \circ (B_n \cdot (up+p)) \circ \cdots \circ (B_n \cdot (up+h_n)).
\]

where \( h_n - 1 = l_n + t - 1 \mod p \).

For each \( 0 \leq u < m \), let \( F_n^u \) be the set of \( x \in X \) having

\[
[(B_n \cdot (up+p)) \circ (B_n \cdot (up+1)) \circ \cdots \circ (B_n \cdot (up+p-1))]^2
\]

for a \( G \)-name of length \( 2pq_n \). We have \( F_n^u \subseteq E_n^{up+p} \) and

\[
\mu(F_n^u) \geq pk_n/rq_{n+1} = pk_n/mps_nq_n \\
\geq k_n/mp(k_n + 3)q_n \\
= 1/mpq_n - 3/mp(k_n + 3)q_n.
\]

Since \( k_n \to \infty \),

\[
q_n' \mu(F_n^u) \uparrow 1/m,
\]

where \( q_n' = pq_n \). This implies (i) of Definition 4 for the sets \( \{F_n^1, \ldots, F_n^m\} \). Let \( \eta_n = \{T^j F_n^u : 0 \leq j < pq_n, 0 \leq u < m\} \). It follows from (6) that for each \( F_n \in \eta_n \) there exists a unique \( E_n \in \xi_n \) with \( F_n \subseteq E_n \). Thus all the sets in \( \eta_n \) are disjoint and (ii) holds. It also follows from (7) that \( \mu(F_n)/\mu(E_n) \uparrow 1 \) as \( n \to \infty \). This implies (iii) for \( \eta_n \) since \( \xi_n \) satisfies (iii). It follows that \( R \leq m \). \( \blacksquare \)

**THEOREM.** Given a positive integer \( m > 1 \) there exists a minimal uniquely ergodic non-abelian Morse sequence \( x \) with spectral multiplicity \( M = m \).

Of course the classical Morse sequence has simple spectrum (cf. [J]) so we get arbitrary finite multiplicity in the class of generalized Morse sequences based on arbitrary finite groups.

**Comments.** Although we assumed (3) it was not really necessary. It is easy to see for \( n \in N_3 \) we can take

\[
A_n = (1, 2) \times (1, 3) \times (1, 5) \times \cdots \times (1, (2^k \mod m) + 1).
\]

for \( k > n \). Similarly it is possible to replace \( A_n \), for \( n \) in \( N_1 \) and \( N_2 \), by products of arbitrarily short blocks. It then follows from the associativity of the block product that we can achieve (3) by pre-multiplying sequences of adjacent short blocks. This shows that
the Theorem holds without any conditions on the block length. In particular, the adding
machine factor $T_0$ can be arbitrary.

By using a combination of the arguments in [L2] and [R1] one can show more. For
any given adding machine $T_0$, and $G = \mathbb{Z}/m \times \mathbb{Z}/p$, the “generic” $G$ extension $T$ of $T_0$
is isomorphic to a non-abelian Morse sequence based on $G$, which has multiplicity $m$.
Moreover, by using different groups $G$ to construct Morse sequences, it is possible to obtain
more exotic spectral results like those in [R3] (cf. also [R4]). Finally we note that the
approximation theory in [KaSt] shows that the continuous part of the spectrum in all of
these examples is singular.

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