Problem 1

STRATEGY

KNOWN

Resistance of a complete spherical shell:

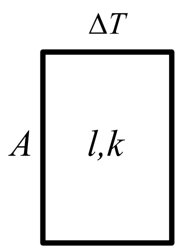
$R_{sphere} = (r_o - r_i) / (4\pi r_o r_i k)$

Inner and outer radii r_i and r_o ,

SOLUTION

<u>Part 1: Resistance of a hemispherical shell:</u>

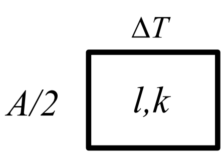
To calculate the resistance of the hemisphere, use analogy with slab:



The heat flux through a slab is:

$$q'' = \frac{\Delta T}{R} = \frac{\Delta T}{\frac{l}{kA}}$$

Now, for the slab of same length, *l,* ***<u>half</u>*** the area *(A/2)* and the same heat flux:



$$q'' = \frac{\Delta T}{\frac{l}{k\frac{A}{2}}} = \frac{\Delta T}{R'} = \frac{\Delta T}{2R}$$

Therefore, for half the area, the resistance is doubled

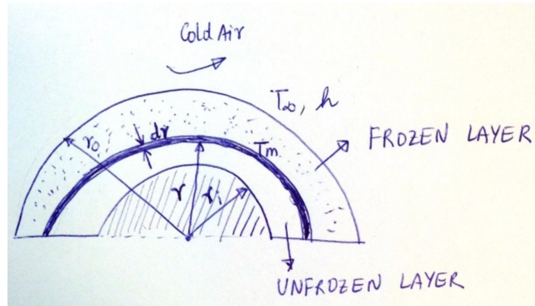
$$R' = 2R$$

Thus, the resistance of the hemisphere twice of that of the complete sphere:

$$R_{hemi-sphere} = 2R_{sphere} = \frac{r_o - r_i}{2\pi r_o r_i k} \qquad \text{(Ans.)}$$

Part 2: Freezing time for hemispherical tissue region:

Parameters assumed:



T_m = Freezing temperature (°C)

T_∞ = Cold air temperature (°C)

ΔH_f = latent heat of fusion (J/kg)

h = heat transfer coefficient (W/m²·K)

ρ = density (kg/m³)

k = thermal conductivity of the frozen layer (W/m·K)

Now, heat loss through a frozen layer at a distance *r* from the center (see Figure) for a *convection boundary condition*:

$$q = \frac{T_m - T_\infty}{R_{hemi-sphere} + 1/Ah} = \frac{T_m - T_\infty}{(r_o - r)/(2\pi r_o rk) + 1/2\pi r_o^2 h} \tag{1}$$

Where, $A = 2\pi r_o^2$ is the outer surface area of the hemisphere

Energy rate from the latent heat given off at the freezing front of thickness *dr* (Note that *r* is decreasing so *dr* is negative)

$$q = -\Delta H_f 2\pi r^2 \rho \frac{dr}{dt} \tag{2}$$

Equation (1) and (2) from above and solving for *t*, time to freeze as a function of r:

$$\frac{T_m - T_\infty}{\frac{r_o - r}{2\pi r_o rk} + \frac{1}{2\pi r_o^2 h}} = -\Delta H_f 2\pi r^2 \rho \frac{dr}{dt}$$

$$-\frac{T_m - T_\infty}{\Delta H_f \rho} dt = \left(\frac{r_o - r}{2\pi r_o rk} + \frac{1}{2\pi r_o^2 h} \right) 2\pi r^2 dr$$

$$-\frac{T_m - T_\infty}{\Delta H_f \rho} dt = \left(\frac{r_o - r}{r_o k} \right) rdr + \left(\frac{1}{r_o^2 h} \right) r^2 dr$$

$$-\frac{T_m - T_\infty}{\Delta H_f \rho} \int_0^{t_m} dt = \int_{r_0}^{r_i} \left(\frac{r_o - r}{r_o k} \right) rdr + \int_{r_0}^{r_i} \left(\frac{1}{r_o^2 h} \right) r^2 dr$$

$$-\frac{T_m - T_\infty}{\Delta H_f \rho}\left[t\right]_0^{t_m} = \int_{r_0}^{r_i} \frac{1}{k}\left(r - \frac{r^2}{r_o}\right)dr + \int_{r_0}^{r_i}\left(\frac{r^2}{r_o^2 h}\right)dr$$

$$= \frac{1}{k}\left(\frac{r^2}{2} - \frac{r^3}{3r_o}\right)_{r_0}^{r_i} + \frac{1}{r_o^2 h}\left(\frac{r^3}{3}\right)\Bigg|_{r_0}^{r_i}$$

$$\frac{T_m - T_\infty}{\Delta H_f \rho}\left[t\right]_0^{t_m} = \frac{1}{k}\left(\frac{r^2{}_o - r_i^2}{2} - \left(\frac{r^3{}_o - r_i^3}{3r_o}\right)\right) + \frac{1}{r_o^2 h}\left(\frac{r^3{}_o - r_i^3}{3}\right)$$

$$= \frac{1}{k}\left(\frac{r_o^2}{2} - \frac{r_o^2}{3} - \frac{r_i^2}{2} + \frac{r_i^3}{3r_o}\right) + \frac{1}{3h}\left(r_o - \frac{r_i^3}{r_o^2}\right)$$

$$= \frac{1}{k}\left(\frac{r_o^2}{6} - \frac{r_i^2}{2} + \frac{r_i^3}{3r_o}\right) + \frac{1}{3h}\left(r_o - \frac{r_i^3}{r_o^2}\right)$$

$$t_m = \frac{\Delta H_f \rho}{T_m - T_\infty}\left[\frac{1}{k}\left(\frac{r_o^2}{6} - \frac{r_i^2}{2} + \frac{r_i^3}{3r_o}\right) + \frac{1}{3h}\left(r_o - \frac{r_i^3}{r_o^2}\right)\right] \qquad \text{(Ans.)} \qquad (3)$$

<u>Part 3: Checking for consistency with sphere formula</u>

In (3) above, for $r_i \to 0$ and $h \to \infty$, the equation reduces to:

$$t_m = \frac{\Delta H_f \rho}{k(T_m - T_\infty)}\left(\frac{r_o^2}{6}\right) \qquad \text{(Ans.)} \qquad (4)$$

which is consistent with the formula for freezing a sphere of radius $r_{o.}$

Problem 2

STRATEGY

KNOWN

Surface temperature of humans *T*= 33 °C = 306 K

Infrared thermography operates in the wavelength range $6-14\ \mu\text{m}$

SOLUTION

Maximum energy emitted by the human surface (covering the entire wavelength range):

$$q''_{\max} = \sigma T^4 \qquad (1)$$

where,

$\sigma = 5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4$•

T = 306 K

Substituting in (1) above, $$q''_{\max} = 497.13\ \text{W/m}^2 \qquad (2)$$

Now, fraction of energy between ($\lambda_2 = 14\ \mu\text{m}$) and ($\lambda_1 = 6\ \mu\text{m}$) $= F_{0-\lambda_2 T} - F_{0-\lambda_1 T}$ (3)

$$\lambda_1 T = 1836; \quad F_{0-1836} = 0.045$$
$$\lambda_2 T = 4284; \quad F_{0-4284} = 0.53$$

Thus,

$$F_{net} = F_{0-4284} - F_{0-1836} = 0.53 - 0.045 = 0.485 \qquad (4)$$

Therefore, fraction of energy emitted by the human surface in the wavelength range $6-14\ \mu\text{m}$

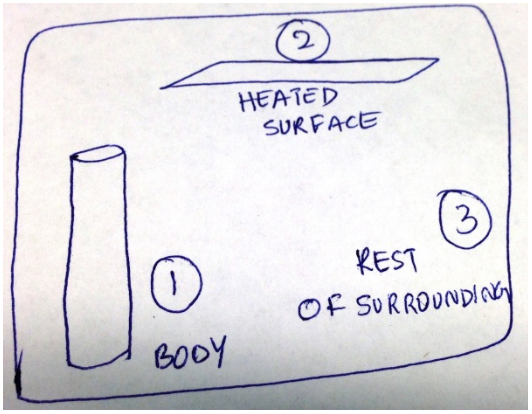
$$q'' = F_{net}q''_{\max} = 241.10\ \text{W/m}^2 \quad \text{(Ans.)} \qquad (5)$$

Problem 3

STRATEGY

KNOWN

There are three radiating surfaces—the human body surface, the heated plate and the rest of the surroundings.
Emissivity of clothed body is 1



ASSUMPTIONS

1) Heated surface and *rest of surroundings* approximated as a black body
2) Person is assumed as vertical cylinder

SOLUTION

Part 1: Net radiative exchange between the person (1) and the heated surface (2)

$$q_{12} = \sigma A_1 F_{12}(T_p^{\,4} - T_h^4) \quad (1)$$

Where,

A_1 = surface area of the person (m^2)

F_{12} = View Factor from person (1) to heated surface (2)

T_p = person surface temperature (K)

T_h = heated surface temperature (K)

σ = Stefans-Boltzman Constant (5.67×10^{-8} W/m$^2 \cdot$ K^4)

Part 2: Relationship between size x of the heated surface and air velocity over the person's surface

For the person:

$$\textbf{IN – OUT + GENERATION = STORAGE} \quad (2)$$

IN: *none*

OUT: $-\sigma A_1 F_{12}(T_p^{\,4} - T_h^4) - \sigma A_1 F_{13}(T_p^{\,4} - T_a^4) - hA_1(T_p - T_b)$

GENERATION: $Q_s A_1$

STORAGE: *none (steady state)*

Plugging in (2) above

$$\underbrace{0}_{\text{IN}}-\underbrace{\sigma A_1F_{12}(T_p^{\,4}-T_h^4)-\sigma A_1F_{13}(T_p^{\,4}-T_a^4)-hA_1(T_p\text{ - }T_b)}_{\text{OUT}}+\underbrace{\mathrm{Q}_s\,A_1}_{\text{GEN.}}=0 \qquad (3)$$

Now, information on air velocity u_∞ is embedded in the heat transfer coefficient, h.

$$Nu_D=\frac{hD}{k}=BRe_D^nPr^{1/3}$$

$$=B\left(\frac{\rho u_\infty D}{\mu}\right)^n Pr^{1/3} \qquad (4a)$$

From above,

$$h=\frac{u_\infty^n}{\left(\dfrac{1}{B\,\mathrm{Pr}^{1/3}}\right)\left(\dfrac{\mu}{\rho D}\right)^n\left(\dfrac{D}{k}\right)} \qquad (4b)$$

The information on length, x is present in the view factor from the person to the heated plate, F_{12}. Writing h as a function of F_{12}:

$$-\sigma\cancel{A_1}F_{12}(T_p^{\,4}-T_h^4)-\sigma\cancel{A_1}F_{13}(T_p^{\,4}-T_a^4)+\mathrm{Q}_s\,\cancel{A_1}=h\cancel{A_1}(T_p\text{ - }T_b) \qquad (5)$$

$$h=\frac{\mathrm{Q}_s-F_{12}(T_p^{\,4}-T_h^4)-\sigma F_{13}(T_p^{\,4}-T_a^4)}{(T_p\text{ - }T_b)} \qquad (6)$$

Substituting for h in (6) from (4b) and rearranging:

$$u_\infty^n=\left(\frac{\mathrm{Q}_s-F_{12}(T_p^{\,4}-T_h^4)-\sigma F_{13}(T_p^{\,4}-T_a^4)}{(T_p\text{ - }T_b)}\right)\left(\frac{1}{B\,\mathrm{Pr}^{1/3}}\right)\left(\frac{\mu}{\rho D}\right)^n\left(\frac{D}{k}\right) \qquad \text{(Ans.)}$$

Part 3: Convective heat loss from the body

In order to obtain the convective heat loss from the body, we first need to obtain the convective heat transfer coefficient, h.

The person assumed as a vertical cylinder, therefore, the following correlation to calculate h holds good (for both laminar and turbulent flows):

$$Nu_D=BRe_D^nPr^{1/3} \qquad (7)$$

Note, properties of air in above Eqn. (7) need to be obtained at the film temperature, T_f

$$T_f = \frac{T_p + T_b}{2} \tag{8}$$

GIVEN PARAMETERS

$D = 0.45$ m

$v = 0.1$ m/s

$T_p = 33$ °C = 306 K

$T_b = 0$°C = 273 K

Thus, $T_f = 16.5$ °C = 289.5 K

Properties of air at T_f:

$\mu = 1.7985 \times 10^{-5}$ Pa·s

$\rho = 1.2177$ kg/m^3

$k = 0.02547$ W/m·K

$\Pr = 0.710$

Reynolds No., $Re_D = \dfrac{\rho v D}{\mu}$ = 3046.8.

Flow is Laminar. Therefore, from table, $B = 0.683$ and $n = 0.466$

$$Nu_D = \frac{hD}{k} = 0.683(\mathrm{Re}_D)^{0.466}(Pr)^{1/3}$$

$$h = \frac{k}{D} \times 0.683(\mathrm{Re}_D)^{0.466}(Pr)^{1/3} = \frac{0.02547}{0.45} \times 0.683 \times (3046.8)^{0.466} \times (0.710)^{1/3}$$

$h = 1.45$ W/m^2·K

Convective loss,

$$q''_{conv} = h(T_p - T_h)$$
$$= 1.45(306 - 273) = 47.85 \text{ W/m}^2 \qquad \text{(Ans.)}$$

<u>Part 4: Length, x of the heated surface</u>

To calculate the length, x, we need to first find the view Factor F_{12}. Then, use the chart that relates F_{12} with X and Y to obtain the length, x.

From Eqn. (3) above:

$$\sigma A_1 F_{12}(T_p^{\,4} - T_h^4) + \sigma A_1 F_{13}(T_p^{\,4} - T_a^4) + hA_1(T_p - T_b) = Q_s\, A_1 \quad (9)$$

$$\sigma F_{12}(T_p^{\,4} - T_h^4) + \sigma F_{13}(T_p^{\,4} - T_a^4) + h(T_p - T_b) = Q_s \quad (10)$$

Using the following relationship and substituting for F_{13} in Eqn. (9) above:

$$\begin{aligned} F_{12} + F_{13} &= 1 \\ F_{13} &= 1 - F_{12} \end{aligned} \quad (11)$$

$$\sigma F_{12}(T_p^{\,4} - T_h^4) + \sigma(1 - F_{12})(T_p^{\,4} - T_a^4) + h(T_p - T_b) = Q_s \quad (12)$$

Rearranging Eqn. (11) to obtain an explicit expression for F_{12}

$$F_{12} = \frac{Q_s - h(T_p - T_b) - \sigma(T_p^{\,4} - T_a^4)}{\sigma(T_a^{\,4} - T_h^4)} \quad (13)$$

Note, in Eqn. (12), $q''_{conv} = h(T_p - T_b)$ is the convective loss calculated in ***<u>Part 3</u>*** above

All terms on the R.H.S of Eqn. (13) are known:

$Q_s = 70$ W/m^2

$T_p = 33$ °C = 306 K

$T_b = 0$ °C = 273 K

$T_h = 200$ °C = 473 K

$q''_{conv} = 47.85$ W/m^2 (from above)

Plugging in numbers and solving for F_{12}

$$F_{12} = \frac{Q_s - q''_{conv} - \sigma(T_p^{\,4} - T_a^4)}{\sigma(T_a^{\,4} - T_h^4)} = \frac{70\left[\frac{\text{W}}{\text{m}^2}\right] - 47.85\left[\frac{\text{W}}{\text{m}^2}\right] - 5.67\times10^{-8}\left[\frac{\text{W}}{\text{m}^2\cdot\text{K}^4}\right](306^4 - 273^4)\text{K}^4}{5.67\times10^{-8}\left[\frac{\text{W}}{\text{m}^2\cdot\text{K}^4}\right](273^4 - 473^4)\text{K}^4}$$

Thus, $$F_{12} = 0.063 \quad (14)$$

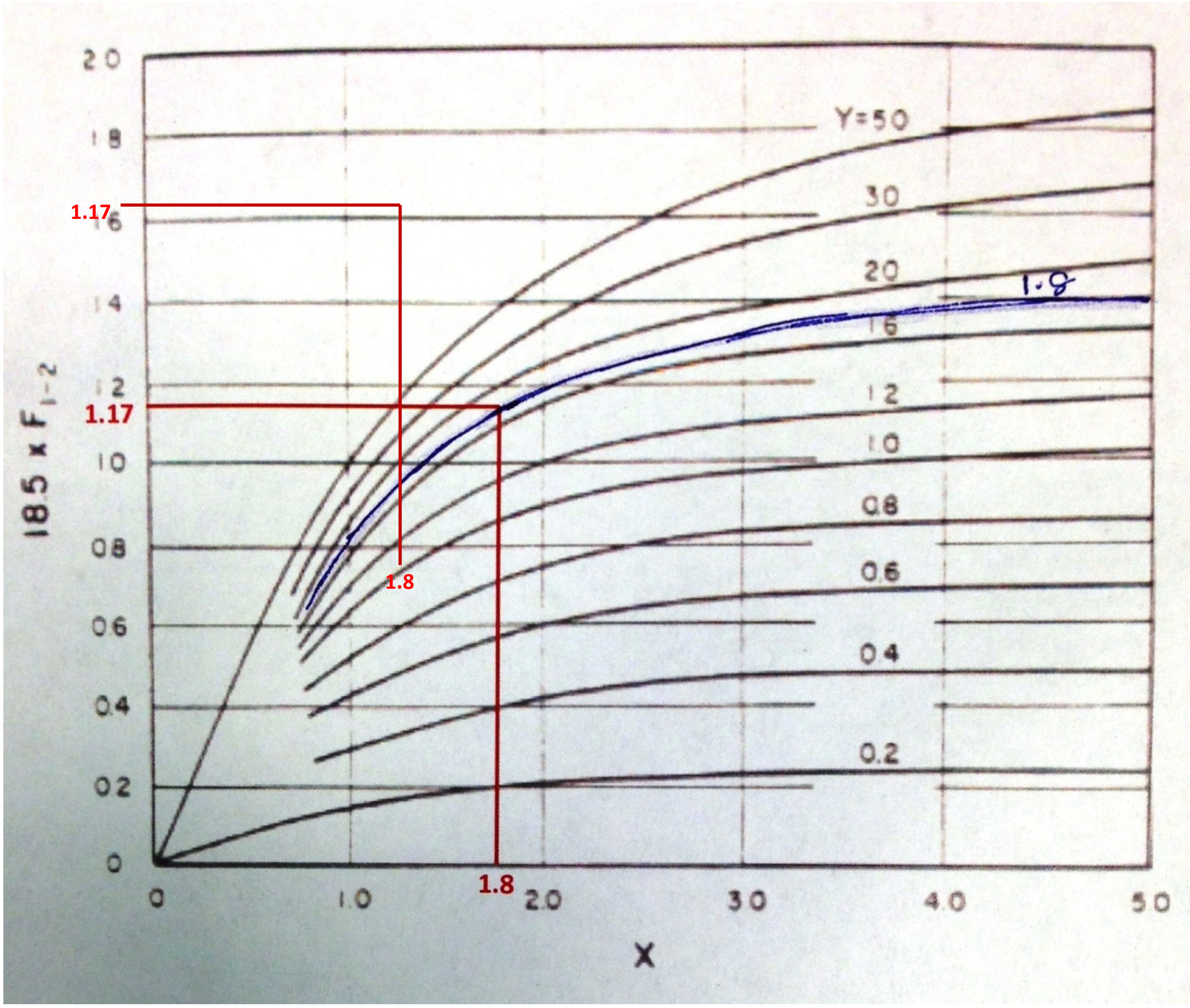
In order to read the chart we need to multiply F_{12} with a factor of 18.5

$$18.5F_{12} = 1.17$$

From the chart, for $18.5F_{12} = 1.17$ and $X = Y$, the value of X corresponds to 1.8 (see chart below)

$$X = \frac{x}{z} = 1.8 \text{ and } z = 2.5 \text{ m}$$

$$x = 1.8z = 1.8 \times 2.5 = 4.5 \text{ m} \qquad \text{(Ans.)}$$



Problem 4

STRATEGY

ASSUMPTION

Assume fish has a mass M g

KNOWN

$t = 84$ days

$c_0 = 0$ μg/g

$\alpha = 0.8$

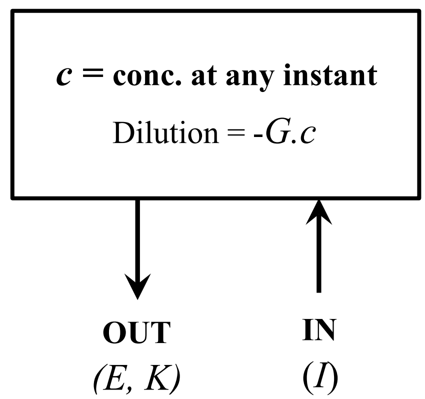
$E = 0.007$ 1/day

$K = 8\times10^{-5}$ 1/day

$I = 0.008$ 1/day

$G = 0.007$ 1/day

$c_d = 0.13$ μg/g



<u>Part 1: Mass Balance for mercury in the fish over time Δt</u>

$$\textbf{IN – OUT + GENERATION = STORAGE} \quad (1)$$

$\text{IN} = M\alpha I c_d$

$\text{OUT} = -M(E+K)c$

$\text{GEN.} = -MGc$

$\text{STORAGE} = M\Delta c$

Here, GENERTION refers to the dilution of mercury concentration due to fish growth

Substituting in (1) above,

$$\left[M\alpha I c_d - M(E+K)c - MGc\right]\Delta t = M\Delta c \quad \text{(Ans.)}$$

<u>ALTERNATIVE METHOD:</u>

Since the fish grows, its mass changes. So, the STORAGE term can be alternatively written as:

$$\begin{aligned}\text{STORAGE} &= \Delta(cM) \\ &= c\Delta M + M\Delta c\end{aligned} \tag{2}$$

The specific growth rate, *G,* is the change in body weight per body weight per time i.e.

$$G = \frac{\Delta M}{M\Delta t} \tag{3}$$

Substituting for *ΔM* from above:

$$\Delta M = GM\Delta t \tag{4}$$

$$\text{STORAGE} = cGM\Delta t + M\Delta c \tag{5}$$

Note, in this approach there is <u>no GENERATION term;</u> IN and OUT remain the same as above:

IN – OUT = STORAGE

$$M\alpha Ic_d\Delta t - M(E+K)c\Delta t = M\Delta c + cMG\Delta t \tag{6}$$

Rearranging Eqn.(6), we get the same expression as above

$$\left[M\alpha Ic_d - M(E+K)c - MGc\right]\Delta t = M\Delta c \qquad \text{(Ans.)}$$

<u>Part 2: Differential equation for the concentration of mercury,c as a function of time, t</u>

$$\left[M\alpha Ic_d - M(E+K)c - MGc\right]\Delta t = M\Delta c$$

$$\frac{\Delta c}{\Delta t} = \alpha Ic_d - (E+K+G)c$$

$$\frac{dc}{dt} = \alpha Ic_d - (E+K+G)c \qquad \text{(Ans.)}$$

<u>Part 3: Solve the Differential equation for the concentration of mercury,c as a function of time, t</u>

Let, $A = \alpha Ic_d$ and $B = E+K+G$

Therefore, the differential equation in terms of *A* and *B*:

$$\frac{dc}{dt} = A - Bc \tag{7}$$

$$\frac{dc}{A-Bc} = dt$$

Integrating both sides to solve for *c(t):*

$$\int_{c_0}^{c(t)} \frac{dc}{A - Bc} = \int_0^t dt$$

$$-\frac{1}{B}[ln(A - Bc)]_{c_0}^{c(t)} = t$$

$$-\frac{1}{B} ln\left(\frac{A - Bc(t)}{A - Bc_0}\right) = t$$

$$\frac{A - Bc(t)}{A - Bc_0} = e^{-Bt}$$

$$c(t) = \frac{A - (A - Bc_0)\,\mathrm{e}^{-Bt}}{B}$$

Plugging in the values of *A* and *B*:

$$c(t) = \frac{\alpha I c_d - [\alpha I c_d - (E + K + G)c_0]\,\mathrm{e}^{-(E+K+G)t}}{E + K + G} \qquad \text{(Ans.)}$$

<u>Part 4: Mercury concentration inside fish after 84 days</u>

From above,

$$c(t) = \frac{\alpha I c_d - [\alpha I c_d - (E + K + G)c_0]\,\mathrm{e}^{-(E+K+G)t}}{E + K + G} \qquad (8)$$

All values on the R.H.S are known. For in initial conc., $c_0 = 0$, Eqn. (8) reduces to:

$$c(t) = \frac{\alpha I c_d (1 - \mathrm{e}^{-(E+K+G)t})}{E + K + G} \qquad (9)$$

Plugging in the values of α, E, K, G, I, c_d in Eqn. (9) and solving for time $t = 84$ days:

$$c(t = 84) = \frac{(0.8)\left(0.008\frac{1}{\text{day}}\right)\left(0.13\frac{\mu\text{g}}{\text{g}}\right)\left(1 - \mathrm{e}^{-\left(0.007\frac{1}{\text{day}} + 0.00008\frac{1}{\text{day}} + 0.007\frac{1}{\text{day}}\right)84\text{ day}}\right)}{\left(0.007\frac{1}{\text{day}} + 0.00008\frac{1}{\text{day}} + 0.007\frac{1}{\text{day}}\right)} \qquad (10)$$

Therefore,

$$c(t = 84\text{ days}) = 0.041\ \mu\text{g/g} \qquad \text{(Ans.)}$$