Problem 4.8.24

KNOWN: Newborn head cooled to protect brain from damage. Initial and final (desired) temperature of the center of the head is known.

FIND: Environmental (air) temperature so that the center of the head reaches a temperature of 35 °C.

SCHEMATIC AND GIVEN DATA

r = 4 cm
k = 0.5 W/mK
h = 15 W/m^{2}K
Metabolic heat production = 4500 W/m^3
Positive heat contribution by blood flow = 5500 W/m^3
Normal temperature at the center of the head = 37°C
Final temperature at the center of the head = 35 °C

STRATEGY: We use the governing equation for heat conduction in a spherical system at steady state assuming radial symmetry.

ASSUMPTIONS:

Radial symmetry
Steady state

SOLUTION:

For a sphere at steady state, with radial symmetry and internal heat generation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{Q}{k}$$
$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{Q}{k}r^2$$

Integrating,

$$r^2\frac{\partial T}{\partial r} = -\frac{Q}{3k}r^3 + c_1$$

At r = 0, $\frac{\partial T}{\partial r}$ = 0 (radial symmetry)
Therefore, $c_1 = 0$.

$$\frac{\partial T}{\partial r} = -\frac{Q}{3k}r$$

Integrating again,

$$T = -\frac{Q}{6k}r^2 + c_2$$

At r = R, T = T_s

$$T_s = -\frac{Q}{6k}R^2 + c_2$$

Using the boundary condition $-k\frac{\partial T}{\partial r} = h(T_s - T_\infty)$

$$-k\left(-\frac{Q}{3k}R\right) = h\left(-\frac{Q}{6k}R^2 + c_2 - T_\infty\right)$$

Problem 4.8.24

$$\frac{Q}{3h}R = -\frac{Q}{6k}R^2 + c_2 - T_\infty$$

$$c_2 = \frac{Q}{3h}R + \frac{Q}{6k}R^2 + T_\infty$$

$$T = -\frac{Q}{6k}r^2 + \frac{Q}{3h}R + \frac{Q}{6k}R^2 + T_\infty \qquad \dots(A)$$

At r=0, T=T_0

$$T_0 = \frac{Q}{3h}R + \frac{Q}{6k}R^2 + T_\infty$$

$$T_0 - T_\infty = \frac{Q}{3h}R\left(\frac{Rh}{2k} + 1\right)$$

$$T_0 - T_\infty = \frac{4500\,\frac{\mathrm{W}}{\mathrm{m}^3} + 5500\,\frac{\mathrm{W}}{\mathrm{m}^3}}{3 \times 15\ \mathrm{W/m^2K}} \times 0.04\ \mathrm{m} \times \left(\frac{0.04\ \mathrm{m} \times 15\ \mathrm{W/m^2K}}{2 \times 0.5\ \mathrm{W/mK}} + 1\right)$$

$$T_0 - T_\infty = 14.22\ \mathrm{K}$$

$$T_\infty = 35\ °\mathrm{C} - 14.22\ °\mathrm{C} = 20.78\ °\mathrm{C}$$

COMMENTS

If we want to check the surface temperature we would use equation (A) for r=R:

$$T_S = -\frac{Q}{6k}R^2 + \frac{Q}{3h}R + \frac{Q}{6k}R^2 + T_\infty$$

$$T_S = \frac{Q}{3h}R + T_\infty$$

$$T_S = \frac{4500\,\frac{\mathrm{W}}{\mathrm{m}^3} + 5500\,\frac{\mathrm{W}}{\mathrm{m}^3}}{3 \times 15\,\frac{\mathrm{W}}{\mathrm{m^2K}}} \times 0.04\ \mathrm{m} + 20.78\ °\mathrm{C} = 29.67\ °\mathrm{C}$$

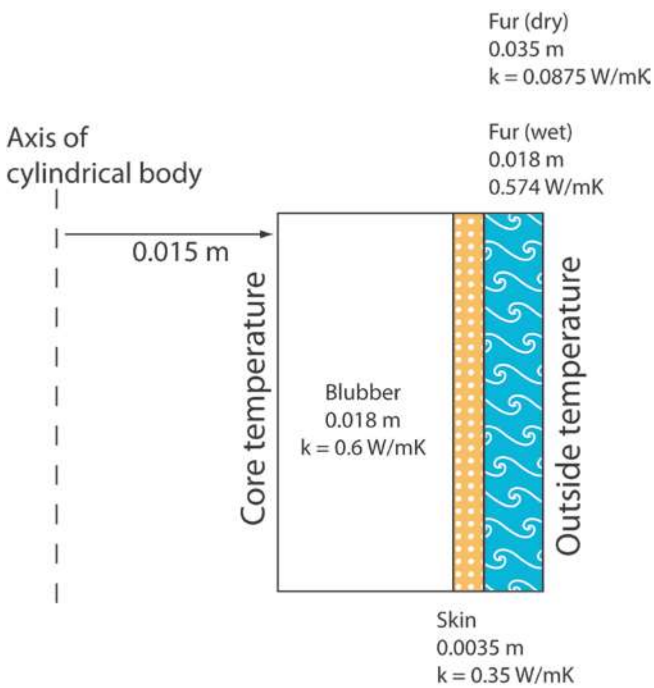
Problem 4.8.32

Problem 4

KNOWN:
Dimensions of baby seal and thermal conductivity of each layer. The driving force for heat.

FIND:
Resistance in each layer and the fraction of resistance provided by either a dry or wet fur layer. Then, find the heat flow through the dry and wet baby seal. Finally, find the ration of resistance.

SCHEMATIC AND GIVEN DATA



STRATEGY:
Calculate the total resistance (in cylindrical geometry) when there is dry fur and when there is wet fur. Calculate the fraction of resistance for dry fur and wet fur. Calculate the heat flow for a dry pup—this is equal to the metabolic heat generated at steady state. Ratio of two heat flows (dry and wet) at same temperature difference is the reciprocal of the ratio of resistances.

ASSUMPTIONS:

None

SOLUTION:

1)

Dry fur:

R_i dry fur = 0.015+0.018+0.0035 = 0.0365 m
R_o dry fur = 0.0365 + 0.035 = 0.0715 m
Wet fur:
R_i wet fur = 0.015+0.018+0.0035 = 0.0365 m
R_o wet fur = 0.0365 + 0.018 = 0.0545 m
Skin:
R_i skin = 0.015+0.018 = 0.033 m
R_o skin = 0.033 + 0.0035 = 0.0365 m
Blubber:
R_i blubber = 0.015
R_o blubber = 0.015 + 0.018 = 0.033 m

$$R_{TotDry} = \frac{\ln\frac{0.033}{0.015}}{2\pi \times 0.6 \times 0.9} + \frac{\ln\frac{0.0365}{0.033}}{2\pi \times 0.35 \times 0.9} + \frac{\ln\frac{0.0715}{0.0365}}{2\pi \times 0.0875 \times 0.9} = 1.64\frac{K}{W}$$

$$R_{Totwet} = \frac{\ln\frac{0.033}{0.015}}{2\pi \times 0.6 \times 0.9} + \frac{\ln\frac{0.0365}{0.033}}{2\pi \times 0.35 \times 0.9} + \frac{\ln\frac{0.0545}{0.0365}}{2\pi \times 0.574 \times 0.9} = 0.403\frac{K}{W}$$

$$R_{Dry} = \frac{\ln\frac{0.0715}{0.0365}}{2\pi \times 0.0875 \times 0.9} = 1.36\frac{K}{W}$$

$$R_{wet} = \frac{\ln\frac{0.0545}{0.0365}}{2\pi \times 0.574 \times 0.9} = 0.124\frac{K}{W}$$

1)
Fraction of resistance

$$\frac{R_{Dry}}{R_{TotDry}} = 82.9\%$$

2)

Fraction of resistance when wet

$$\frac{R_{wet}}{R_{TotWet}} = 30.8\%$$

The water decreases the thickness and increases the conductivity therefore increasing the rate at which the baby loses heat.

3)

$$q''_D = \frac{37-4}{R_{TotDry}} = 20.1W \qquad q''_W = \frac{37-4}{R_{Totwet}} = 81.9W$$

$$Q = \frac{20.1}{\pi R^2 L} = \frac{20.1}{\pi 0.015^2 0.9} = 31595\frac{W}{m^3} \quad Q_W = \frac{81.9}{\pi R^2 L} = \frac{81.9}{\pi 0.015^2 0.9} = 128593\frac{W}{m^3}$$

4) $\frac{Q_w}{Q_D} = 4.07$

Problem 4.8.33

Known:
Steady state heat transfer in concentric spheres. Two conductive layers and convective transport from ambient.

Find:
Temperature for a snow and cardboard model and for a U-Vacua model only. Find all three heat flows. The overall thermal resistance and the heat flow from snow to ambient. Percent of snow that melted.

Schematic and Given Data:
$k_c = 0.067$ W/mK
$k_U = 0.00070$ W/mK
$R_i = 0.5$ m
$R_m = 0.6$ m
$R_o = 0.62$ m

Assumptions:
Steady State. Snow is at constant temperature.

Strategy:
Going down the decision tree, this is transient heat transfer in a spherical shell. Since the problem is asking to derive the equation for temperature starting from governing equation, we would proceed to do so as opposed to writing the solution if it is already available. Start with the heat equation and cancel transient, convective, and heat generation term. Apply boundary conditions and solve for temperature. Then, take the derivative of temperature to obtain heat flow. Rearrange heat flow in terms of temperature difference and resistance, to find the resistance quantity. Heat flow from the snow to air will be obtained by using the total thermal resistance of the cardboard, insulation and air and the total temperature difference. Calculate the heat gained over 40 hours using heat flow. Calculate mass of snow and find percent that would melt.

Solution:

1)

$$\rho C_p\left(\frac{\partial T}{\partial t}+v\cdot\nabla T\right)=k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\right]+Q$$

$$\frac{\partial T}{\partial t}=0 \quad \text{steady state}$$

$$v\cdot\nabla T=0 \quad \text{no convection}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}=0 \quad \text{only radial variance}$$

$$Q=0 \quad \text{no generation}$$

$$0 = k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right]$$

$$T(r = R_i) = T_i$$

$$T(r = R_m) = T_m$$

2)

$$0 = k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right]$$

$$0 = \frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

$$c_1 = r^2\frac{\partial T}{\partial r} \Rightarrow \frac{c_1}{r^2} = \frac{\partial T}{\partial r}$$

$$T = c_2 - \frac{c_1}{r}$$

$$T_i = c_2 - \frac{c_1}{R_i}$$

$$T_m = c_2 - \frac{c_1}{R_m}$$

$$T_m - T_i = c_1\left[\frac{1}{R_i} - \frac{1}{R_m}\right]$$

$$c_1 = (T_m - T_i)\left(\frac{R_m R_i}{R_m - R_i}\right)$$

$$R_i T_i = R_i c_2 - c_1$$

$$R_m T_m = R_m c_2 - c_1$$

$$c_2 = \frac{R_m T_m - R_i T_i}{R_m - R_i}$$

$$T = \frac{R_m T_m - R_i T_i}{R_m - R_i} - \frac{(T_m - T_i)}{r}\left(\frac{R_m R_i}{R_m - R_i}\right)$$

Conductive flow is defined as:

$$q_r = -kA\frac{dT}{dr}$$

$$q_r = -kA\frac{dT}{dr} = -k\ (4\pi r^2)\left[\frac{(T_m - T_i)}{r^2}\left(\frac{R_m R_i}{R_m - R_i}\right)\right] = -4\pi k\ (T_m - T_i)\left(\frac{R_m R_i}{R_m - R_i}\right)$$

3) Writing the heat flow in terms of temperature difference and resistance,

$$q_r = \frac{T_i - T_m}{\dfrac{R_m - R_i}{4\pi k_c R_m R_i}}$$

4)
The resistances between snow and the ambient air are those of the cardboard, u-vacua, and outside convective

$$R_{card} = \frac{R_m - R_i}{4\pi k_c R_m R_i} \quad R_{U-Vacua} = \frac{R_0 - R_m}{4\pi k_U R_0 R_m} \quad R_{convective} = \frac{1}{h\left(4\pi R_0^2\right)}$$

There three resistances are in series,

$$\sum R = R_{card} + R_{U-Vacua} + R_{amb} = \frac{R_m - R_i}{4\pi k_c R_m R_i} + \frac{R_0 - R_m}{4\pi k_U R_0 R_m} + \frac{1}{h\left(4\pi R_0^2\right)} = 6.53$$

$$q_r = \frac{T_i - T_\infty}{\sum R} = \frac{43.3 - 0^\circ C}{6.53 \, k/W} = 6.64\, W$$

5)
Over the course of 40 hours, $6.64W$ equals:

$$\left(40 \times 3600 s\right) \times 6.64 \times 10^{-3}\, kW = 955.7\, kJ$$

Converting that to kg of melted ice.

$$\frac{955.7\, kJ}{333 \frac{kJ}{kg}} = 2.87\, kg$$

Calculating the amount of ice the snowman is made of:

$$\frac{4}{3}\pi R_i^3 \times 700 \frac{kg}{m^3} = \frac{4}{3}\pi 0.5^3 \times 700 \frac{kg}{m^3} = 367 kg$$

We melted only $\left(2.87/367\right) \times 100 = 0.8\%$