Instructions: Solutions to different problems should go on separate pages. Write down your name and
the problem number on the upper left corner of all pages that you submit for grading. Show your work
and justify all of your steps. Submit only what you want to be graded; no blank paper or scratch paper.

NO CALCULATORS
2 hours

1. The coins in Merryland all have different integer values: there is a single 1 cent coin, a single 2 cent
coin, etc. What is the largest number of coins that a resident of Merryland can have if we know that
their total value does not exceed 2021 cents?

2. For every positive integer \( k \) let

\[
a_k = \left( \frac{\sqrt{k+1}}{k} + \frac{\sqrt{k+1}}{k} - \frac{1}{k} - \sqrt{\frac{1}{k}} \right)
\]

Evaluate the product \( a_4a_5 \cdots a_{99} \). Your answer must be as simple as possible.

3. Prove that for every positive integer \( n \) there is a permutation \( a_1, a_2, \ldots, a_n \) of \( 1, 2, \ldots, n \) for which
\( j + a_j \) is a power of 2 for every \( j = 1, 2, \ldots, n \).

4. Each point of the 3-dimensional space is colored one of five different colors: blue, green, orange, red,
or yellow, and all five colors are used at least once. Show that there exists a plane somewhere in
space which contains four points, no two of which have the same color.

5. Suppose \( a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \) are real numbers. Let \( C_n \) be the union of \( n \) intervals as
below:

\[
C_n = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_n, b_n].
\]

We say \( C_n \) is minimal if there is a subset \( W \) of real numbers \( \mathbb{R} \) for which both of the following hold:

(a) Every real number \( r \) can be written as \( r = c + w \) for some \( c \) in \( C_n \) and some \( w \) in \( W \); and

(b) If \( D \) is a subset of \( C_n \) for which every real number \( r \) can be written as \( r = d + w \) for some \( d \) in
\( D \) and some \( w \) in \( W \), then \( D = C_n \).

(i) Prove that every interval \( C_1 = [a_1, b_1] \) is minimal.

(ii) Prove that for every positive integer \( n \), the set \( C_n \) is minimal.