2002 SOLUTIONS, PART I

Problem 1. By inspection $z < 0$, $0 < x < 1$, and $y > 1$. Thus, $z < x < y$. Answer: e.

Problem 2. If $S$ is my son’s age now then $3S-5=4(S-5)$. Thus $S=15$. Answer: b.

Problem 3. The distance between $2x$ and $4$ is at most 6, so the distance between $x$ and 2 is at most 3. So $-1 < x < 5$. Answer: d.

Problem 4. The equation of the line containing the two points is $y=2x-1$. Answer: c.

Problem 5. Let $S=amount$ Snow White earned. Then $S+7\cdot 20=8(S-3.5)$. So $7S=168$, hence $S=$24. Answer: c.

Problem 6. Setting the two equations equal and simplifying gives $x^2-4x+3=0$. So $x=1$ and 3, which implies $y=2$ and 0. Answer: a.

Problem 7. $A=pi\cdot a^2$, $B=pi\cdot b^2$, and $C=pi\cdot c^2$. Since $c^2=a^2 + b^2$, $C=A+B$. Answer: a.

Problem 8. $log_2(2)=1$ and $log_3(9)=2$. Thus $log_4(x)=3$, so $x=4^3$. Answer: e.

Problem 9. Since 7 Muppets eat 7 cookies in 7 minutes, each Muppet eats a cookie in 7 minutes. So each of the 14 Muppets can eat 2 cookies in 14 minutes, for a total of 28 cookies. Answer: c.

Problem 10. $W(n)$ is the maximal number of wires that can be stretched between $n$ people equally spaced around a circle without any crossings. First, note that the requirement that the people be equally spaced is a red herring - $W(n)$ is also equal to the number non-intersecting wires between $n$ people located anywhere on the circle. We claim that $W(n)=2n-3$ for all $n \geq 2$. Clearly, $W(2)=1$ and $W(3)=3$. Assume $n > 3$ and $W(k)=2k-3$ for all $2 < k < n$. We argue that $W(n)=2n-3$. First, suppose we have any maximal arrangement of wires between $n-1$ people. By assumption, such an arrangement would have $W(n-1)+3=2n-5$ wires. If we add a new person between any two adjacent people, then we clearly can add two new wires: one between the new person and his left-hand and right-hand neighbors. Thus, $W(n)$ is at least $2n-3$. To see that we cannot do better, suppose we have any arrangement of wires between $n$ people. If there are only wires between adjacent people, then there would be only $n$ wires, which is less than $2n-3$, since $n > 3$. So assume that there is a wire between two non-adjacent people, call them A and B. This wire splits the $n$ people into two groups of people: one has size $2 < k < n$, while the other has size $n-k+2$. This is the $n-k+2$-2min number of wires in the original arrangement is at most $W(k)=2k-3$, which is at most $2(2k)+W(n-k)+2=2n-3$. [The -1 term is present since the wire between A and B is included in both subarrangements.] Answer: c.
Problem 22. Let r = speed of the rabbit, let c = speed of the train, t = total elapsed time and let d = distance travelled by the rabbit. Clearly, d = rt and, since the train moves a total distance of 1 mile, t = 1/c, so d = r/c and r = cd. As well, the total time 1/c is equal to 1/(r-c) [the time the rabbit is moving with the train] + 1/(r+c) [the time the rabbit is moving opposite the train]. Substituting cd for r in this equation yields 1/c = 1/(cd-c) + 1/(cd+c). Multiplying both sides by c gives 1 = 1/(d-1) + 1/(d+1), so (d-1)(d+1) = 2d. Thus, d = (1 + 2\sqrt{1/2})/2. Answer: a.

Problem 23. Suppose a and c are integers and there is a right triangle with side lengths a and 17 and hypotenuse c. Then \(c^2 - a^2 = 17^2\). So (c-a)(c+a) is equal to a prime squared. Since a is not 0, it must be that c-a = 1 and c+a = 17^2 = 289. So a = 144 and c = 145 is the only possibility. Answer: e.

Problem 24. Let N = number of days until all 7 dwarfs visit. Then N is the smallest integer that is divisible by 1, 2, ..., 7. So N = 3 · 4 · 5 · 7 = 420. In these 420 days, Dwarf #1 visits 420 times, #2 visits 210 times, #3 visits 140 times, #4 visits 105 times, #5 visits 84 times, #6 visits 70 times, and #7 visits 60 times. Answer: e.

Problem 25. First, it is possible to have 16 participants (their scores could be 000, 011, 022, 033, 101, 112, 123, 130, 202, 213, 220, 231, 303, 310, 321, 332). The general pattern is that the scores are of the form (a,b,c) where a and b are anything (i.e., 16 possibilities) and c = a+b (mod 4). So, for any two participants, if their scores are the same on one of the first two problems but are different on the other, then their score on the third problem will also be different. To see that more than 16 participants is not possible, suppose that there were at least 17 participants. Since there are only 4 possible scores on the first question, there would be a group of at least 5 participants that got the same score on the first question. But then, since there are only 4 answers for the second question, at least two of these 5 would have the same score on the second question. Thus, this pair would have the same score on at least two of the questions, contradiction. Answer: d.