A new way to obtain Bell inequalities

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By requiring of any local realistic theory that the probability of a contradiction of the type considered by Greenberger, Horne, Zeilinger, and Mermin is equal to zero, we obtain a Bell inequality that is violated by the quantum mechanical predictions for spin measurements on a two-particle singlet state.

1. Introduction

Bell proved in 1965 [1] that nonlocality is an unavoidable property of quantum theory. By assuming locality, he derived a set of inequalities which, he then showed, are violated by the predictions of quantum theory. Since then a number of different Bell inequalities have been discovered, the most important of these are the Clauser, Horne, Shimony, and Holt (CHSH) inequalities [2] derived in 1969 and the Clauser and Horne inequalities [3] derived in 1974. Whilst all of these inequalities demonstrate the contradiction between quantum mechanics and local realism, an intuitive understanding of the contradiction is lost in the mathematics of their proofs. For this reason the recent demonstration of a more direct contradiction between quantum mechanics and local realism by Greenberger, Horne, and Zeilinger (GHZ) [4] and Mermin [5] (see also ref. [6]) where inequalities are not necessary has caused much interest. It is not only that these contradictions do not involve inequalities that makes them appealing but also that they provide an immediate and obvious contradiction between quantum mechanics and local realism. Unfortunately, this approach only works for three or more measurements, whereas we are more used to working with two measurements both from a theoretical and from an experimental standpoint. In this paper we will show that the idea behind the GHZ Mermin proofs can be used to obtain a Bell inequality that can be applied in experiments with only two measurements. In deriving this inequality, we do not lose sight of the origin of the contradiction. We will also show that the CHSH inequality can be obtained by using this method.

2. A new way to obtain a Bell inequality

To derive the inequality consider an experiment in which measurements $A(a)$ and $B(b)$ made simultaneously in separate regions $R_1$ and $R_2$ respectively on a quantum system. The quantities $a$ and $b$ are local variables - they can be set locally in the regions $R_1$ and $R_2$ respectively. The possible values of $A(a)$ and $B(b)$ are $\pm 1$ and $0$ where $0$ corresponds to a null result (when the measurement apparatus has failed to register a result). Let the results of the measurements for the $i$th such experiment be $A'(a)$ and $B'(b)$. The assumption of locality is satisfied here because the result of measuring $A(a)$ in the region $R_1$ does not depend on the choice of local variable, $a$, in the region $R_2$ and similarly the result of measuring $B(b)$ does not depend on $a$. Motivated by the GHZ Mermin contradictions and also by the chained inequalities derived by Braunstein and Caves [7] we consider the following statements.

$^1$ The important consideration in Bell theory is not the number of particles in the quantum system but rather the number of measurements with corresponding local variables that we make.
\[ A'(a_1)B'(b_2) = -1 \quad (s.1) \]
\[ A'(a_3)B'(b_3) = -1 \quad (s.2) \]
\[ A'(a_3)B'(b_4) = -1 \quad (s.3) \]
\[ A'(a_5)B'(b_5) = -1 \quad (s.4) \]
\[ \cdots \]
\[ A'(a_{N-1})B'(b_N) = -1 \quad (s.N-1) \]
\[ A'(a_N)B'(b_N) = +1 \quad (s.N) \]

where \( N \) is even. Each quantity \( A'(a_n) \) and \( B'(b_n) \) appears twice on the l.h.s. Hence, the product of all these equations must be equal to +1 or 0 on the l.h.s. but equal to \(-1\) on the r.h.s. Therefore, if all the statements \((s.1)-(s.N)\) are true then there is a contradiction. If we choose to measure \( A(a) \) then we cannot also measure \( A(a_m) \) for the same value of \( m \) if \( a_m \neq a_n \). Consequently, we cannot actually measure all of the quantities in the statements \((s.1)-(s.N)\). However, by considering the probabilities associated with an ensemble of experiments, we will be able to show that all of the statements \((s.1)-(s.N)\) must be true for some of the experiments (i.e. for some values of \( i \)) when certain conditions are met. Let the probability \( p_A(a, b) \) be equal to the number of experiments for which \( A(a)B(b) = \pm 1 \) divided by the total number of experiments in the limit as the total number of experiments tends to infinity. This probability is given by the predictions of quantum theory. The probabilities for each of the statements \((s.1)-(s.N)\) being true are

\[ p_A^+ = p_A(a_1, b_2) \quad (s.1) \]
\[ p_A^- = p_A(a_3, b_3) \quad (s.2) \]
\[ p_A^+ = p_A(a_3, b_4) \quad (s.3) \]
\[ p_A^- = p_A(a_5, b_5) \quad (s.4) \]
\[ \cdots \]
\[ p_A^{(N-1)} = p_A(a_{N-1}, b_N) \quad (s.N-1) \]
\[ p_A^+ = p_A(a_N, b_N) \quad (s.N) \]

These probabilities are not independent and consequently we cannot write down an equation for the probability that all the statements \((s.1)-(s.N)\) are true. However, we can write down an inequality. If \( P \) is the probability that all the statements \((s.1)-(s.N)\) are true then \( 1-P \) is the probability that one or more of the statements is false. The probability that one or more of the statements is false must be less than or equal to the sum of the probabilities for each individual statement being false. The previous sentence represents a crucial step in the argument and so should be understood. It can be expressed mathematically by

\[ 1-P \leq \sum_{n=1}^{N-1} (1-p_A^-) + (1-p_A^+) \quad (1) \]

This simplifies to

\[ P \leq \sum_{n=1}^{N-1} p_A^- + p_A^+ - (N-1) \quad (2) \]

\( P \) is a statistical probability (the number of runs of the experiment for which \((s.1)-(s.N)\) are true divided by the total number of runs of the experiment). Therefore, if \( P > 0 \) then there will certainly be some runs of the experiment for which all the statements \((s.1)-(s.N)\) are true leading to the previously mentioned contradiction. From inequality (2) we see that \( P > 0 \) if

\[ \sum_{n=1}^{N-1} p_A^- + p_A^+ > N-1 \quad (3) \]

If this inequality is satisfied then there is a contradiction between quantum mechanics and local realism. Usually, in Bell theory, we express the contradiction the other way around, thus reversing the above inequality gives the Bell inequality

\[ \sum_{n=1}^{N-1} p_A^- + p_A^+ \leq N-1 \quad (4) \]

If this inequality is violated by quantum mechanics then there is a contradiction between quantum mechanics and local realism. We will now show (i) that inequality (4) can be violated by quantum mechanics and (ii) that there is a local theory which saturates the inequality.

(i) Consider a pair of spin \( \frac{1}{2} \) particles prepared in the singlet state

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|+\rangle_1 |-_2 - |-\rangle_1 |+\rangle_2) \quad (5) \]

Suppose that these two particles separate, particle 1 moving along the \( z \) axis in the +ve direction and
particle 2 moving along the same axis in the 
-ve direction. Then suppose that measurements of spin are
made, \( A(a) \) on particle 1 and \( B(b) \) on particle 2,
along directions at angles \( a \) and \( b \) respectively to the
\( x \) axis in the \( xy \) plane. The possible results of these
measurements are \( \pm 1 \) in appropriate units. We have
assumed that the detectors are ideal so that there are
no null results. According to quantum mechanics
\[
p'(a, b) = \frac{1}{\pi} \cos(a - b).
\]
(6)

We can choose the angles \( a_1, b_1, a_2, ..., b_N \) to be evenly
spread \([0, 2\pi)\) so that
\[
b_n - a_n = 0.
\]
(7)

\[
b_n - a_{n+1} = \pm \frac{\pi}{N-1}.
\]
(8)

Putting (6) into inequality (4) we get a maximum
violation of the inequality when
\[
\phi = \frac{(N-1)\pi}{N}.
\]
(9)

For \( N = 4 \) the inequality can be written
\[
p_1^+ + p_2^- + p_3^- + p_4^+ < 3.
\]
(10)

Using eq. (6) and putting \( \phi = \frac{3\pi}{4} \) (from (9)) the l.h.s.
of (10) is equal to \( 2 + \sqrt{2} \) which violates (10). As
\( N \to \infty \) the l.h.s. of inequality (4) tends to \( N \). In this
same limit, the r.h.s. of inequality (2) tends to 1 and
therefore \( P < 1 \).

(ii) A local model for the spin experiment de-
scribed above is given by Bell [1] (see also ref. [8], p. 86). Simplified, for the present purpose, this model
assumes that there is a hidden variable, \( \lambda \), which is
an angle in the interval \( 0 \leq \lambda \leq 2\pi \) with a uniform dis-
tribution over all the possible directions, i.e. \( \rho(\lambda) = 1/2\pi \).
We put
\[
A^i(a) = \text{sgn}(a - \lambda^i),
\]
(11)

\[
B^i(b) = \text{sgn}(\lambda^i - b),
\]
(12)

where \( \lambda^i \) is the value of \( \lambda \) in the \( i \)th experiment. The
assumption of locality is satisfied by these choices.
Using (11) and (12) and the fact that \( \lambda \) is evenly
distributed we obtain
\[
p^+(a, b) = \frac{|a - b|}{\pi},
\]
(13)

\[
p^-(a, b) = 1 - \frac{|a - b|}{\pi}.
\]
(14)

Making the same choices for \( a_1, b_2, ..., b_N \) as in eqs.
(7) and (8) we find that the probabilities (13) and
(14) saturate inequality (4).

3. Generalizing the inequality

It is a simple matter to generalize the inequality to an experiment involving \( K \) simultaneous measure-
ments \( A_k(a_k), k = 1 \) to \( K \), in \( K \) separate regions \( R_k \).
We are now using the notation \( A_1(a_1), A_2(a_2), ... \)
rather than the previous notation \( A(a), B(b) \). As be-
fore, the possible results of the measurements are \( \pm 1 \)
and 0. In the \( i \)th experiment the result of the mea-
surement \( A_i(a_i) \) is written \( A^i_k(a_k) \).

Consider \( N \) statements, \( (s'.1)-(s'.N) \), of the form
\[
\prod_{k=1}^{K} A^i_k(a_k) = \pm 1, \quad 1 \leq n \leq n_0,
\]
(15)

where \( n_0 \) is odd and \( N \) is even. Note that with \( K = 2, n_0 = N - 1 \), and appropriately chosen \( a_k^i \)’s these state-
ments are the same as the \((s.n)\) statements. If we
choose the \( a_k^i \)’s so that, for each \( k, \)
\[
\prod_{k=1}^{K} A^i_k(a_k^i) = 1 \quad \text{or} \quad 0
\]
(see below) then we have a contradiction because
the product of all the statements \( (s'.1)-(s'.N) \) gives
a \( +1 \) or \( 0 \) on the l.h.s. and \( -1 \) on the r.h.s. (because
\( n_0 \) is odd). Let \( P^\pm \) be the probability that all the
statements \((s'.1)-(s'.N)\) are true and let \( p^\pm_n(a^n) \)
(shorthand for \( p^\pm_n(a_1^n, a_2^n, ..., a^K_n) \)) be the proba-
bility that
\[
\prod_{k=1}^{K} A^i_k(a_k^i) = \pm 1.
\]

As before, we can write
\[
1 - P^\pm = \sum_{n=n_0+1}^{n_0} [1 - p^\pm_n(a^n)]
+ \sum_{n=n_0+1}^{N} [1 - p^\pm_n(a^n)].
\]
(16)
This simplifies to
\[ P' \geq \frac{n_0}{n-1} \sum_{n=1}^{n_0} p_\uparrow (a^n) + \sum_{n=n_0+1}^{N} p_\uparrow (a^n) - (N-1) . \]  
(17)

Proceeding as before, we obtain the more general Bell inequality
\[ \sum_{n=1}^{n_0} p_\uparrow (a^n) + \sum_{n=n_0+1}^{N} p_\uparrow (a^n) \leq N-1 , \]  
(18)
where \( n_0 \) is odd, \( N \) is even, and the \( a_k^n \)'s are chosen so that the condition (15) is satisfied. The condition (15) will be satisfied if the \( a_k^n \)'s can be grouped in pairs such that, for each \( k \), there exist a set of integers \( n_1^k, n_2^k, ..., n_{N/2}^k \) for which it is possible to write
\[ a_k^{n_1^k} = a_k^{n_2^k}, \quad a_k^{n_3^k} = a_k^{n_4^k}, \quad ..., \quad a_k^{n_{N/2-1}^k} = a_k^{n_{N/2}^k} , \]  
(19)
where \( n_i \neq n_j \) if \( j \neq i \).

4. Obtaining the CHSH inequalities

We will now show that it is possible to obtain the CHSH inequalities by using the method used to obtain inequalities (18) and also that it is possible to obtain inequalities (18) by using the method usually employed in deriving the CHSH inequalities. First we consider an ideal experiment for which \( p_\uparrow + p_\downarrow = 1 \), or expressed differently,
\[ p_\uparrow = 1 - p_\downarrow . \]  
(20)
Substituting this into inequality (18) we obtain
\[ -\left( \frac{n_0}{n-1} \sum_{n=1}^{n_0} p_\uparrow (a^n) + \sum_{n=n_0+1}^{N} p_\uparrow (a^n) \right) \leq -1 . \]  
(21)
Adding (18) and (21) gives
\[ 2-N \leq \sum_{n=1}^{n_0} E(a^n) - \sum_{n=n_0+1}^{N} E(a^n) \leq N-2 . \]  
(22)
where
\[ E(a^n) = p_\uparrow (a^n) - p_\downarrow (a^n) . \]  
(23)
If \( N=4, n_0=3, \) and \( K=2 \) then (22) is the CHSH inequality. This derivation works equally well backwards, that is we could obtain (18) from (22) using (20). However, at this stage we have only considered ideal experiments, i.e. when there are no null results. The requirement that there are no null results can be removed by a simple mathematical trick [9]. Consider the “measurement” \( A'_k (a_k) \) which, for the \( k \)th event, is given by
\[ A'_k (a_k) = A_k (a_k) , \quad \text{if } A_k (a_k) = \pm 1 , \]  
\[ = B'_k (a_k) , \quad \text{if } A_k (a_k) = 0 , \]  
(24)
where \( B'_k (a_k) \) is generated randomly to take values +1 and -1 with equal probability. Let \( p'_{\uparrow,\downarrow,\downarrow} (a^n) \) be the probability that the statement
\[ \prod_{k=1}^{N} A'_k (a_k^n) = \pm 1 \]  
(25)
is true. The probability that one or more of the measurements gives a null result for one run of the experiment is \( 1 - (p_{\uparrow} + p_{\downarrow}) \). Half of the runs of the experiment with one or more null results will give +1 for the quantity in (25) and half will give -1 for the same quantity because the \( B'_k (a_k) \) are generated randomly. Therefore
\[ p_{\uparrow,\downarrow,\downarrow} (a^n) = p_{\uparrow,\downarrow,\downarrow} + \frac{1}{2} \left[ 1 - p_{\uparrow,\downarrow,\downarrow} (a^n) - p_{\uparrow,\downarrow,\downarrow} (a^n) \right] . \]  
(26)
This simplifies to
\[ p_{\uparrow,\downarrow,\downarrow} = \frac{1}{2} (1 + p_{\uparrow} (a^n) - p_{\downarrow} (a^n) ) . \]  
(27)
From (23) and (27) we obtain
\[ E^* (a^n) = p_{\uparrow,\downarrow,\downarrow} (a^n) - p_{\uparrow,\downarrow,\downarrow} (a^n) = E(a^n) . \]  
(28)
The way we have defined \( A'_k (a^n) \) ensures that
\[ p_{\uparrow,\downarrow,\downarrow} (a^n) + p_{\uparrow,\downarrow,\downarrow} (a^n) = 1 \]  
(29)
even in non-ideal experiments. The derivation of the inequality (18) applies equally well if we replace \( A_k (a_k) \) with \( A_k (a_k^n) \) in the statements (s' ; n) and \( p_{\uparrow} \) with \( p_{\uparrow,\downarrow,\downarrow} \) in (16)–(18). Using (29) we can then obtain inequalities (22) by the same method as before where \( E(a^n) \) is replaced by \( E^* (a^n) \). However, from (28), \( E^* (a^n) = E(a^n) \) and therefore inequalities (22) apply as they are written even without ideal measurements. If we reverse this argument then we can obtain inequalities like (18) from inequalities (22) but with \( p_{\uparrow,\downarrow,\downarrow} \)'s instead of \( p_{\uparrow} \)'s. From eq. (26) we see that \( p_{\uparrow,\downarrow,\downarrow} \geq p_{\uparrow} \) and therefore we can obtain the Bell inequality (18) from the CHSH type inequalities (22). In fact the inequalities (22) have not been derived before in this general form. However, it is
clear from refs. [8] and [10] that they could be derived by using the method of CHSH.

5. Conclusion

The derivation of the Bell inequality in this paper is simpler and more intuitive than previous derivations of Bell inequalities. The contradiction between quantum mechanics and local realism can be understood in the following way: quantum mechanics requires that a certain set of statements, \((s', 1) - (s', N)\), must all be true for some runs of the experiment even though, from the local realist point of view, they are clearly contradictory when they are all true.

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References