PRIMARY ARITHMETIC: CHILDREN **INVENTING** THEIR OWN **PROCEDURES**

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In an article that appeared in the *Arithmetic Teacher,* Madell (1985) described findings from a private school in New York City in which children were not taught any algorithms until the end of the third grade. Without algorithms, the children devised their own ways of solving computation problems. Madell's observation of the children's thinking led him to conclude that "children not only *can* but *should* create their own computational algorithms" (p. 20) and that "children can and should do their own thinking" (p. 22). The purpose of the present article is to reiterate Madell's call for reform, with supporting evidence from a public school near Birmingham, Alabama.

Children add and subtract from left to right when allowed to invent.

One of Madell's reasons for saying that children should create their own procedures is that in multidigit addition and subtraction, children "*universally* proceed from left to right" Madell (1985, 21). Two of the examples he gave can be seen in **figure 1.** Readers having trouble understanding these examples should be heartened by Madell's assurance that almost everyone else does, too. The lesson to be learned from our difficulty in understanding children's thinking is that "it is hard to follow the reasoning of others. No wonder so many children ignore the best of explanations of why a particular algorithm works and just follow the rules" (Madell 1985, 21).

Since 1984, at Hall-Kent School in Homewood, Alabama, one of the authors has been developing a primary school arithmetic program based on the theory of Jean Piaget. Piaget's theory ([1967] 1971, [1970] 1972), constructivism, states that logico-mathematical knowledge is a kind of knowledge that each child must create from within, in interaction with the environment, rather than acquire it directly from the environment by internalization. On the basis of this theory, the authors have been refraining from teaching algorithms and, instead, have been encouraging children to invent their own procedures for all four arithmetical operations.

Our observations have confirmed Madell's findings every year. Working on addition and subtraction, children in the first two grades always proceed from left to right if they have not been taught to work from right to left and are, instead, encouraged to invent their own procedures. In subtraction, the authors have seen solutions such as the following, besides the two reported by Madell:

50 − 20 = 30,
30 + 3 = 33,
33 − 4 = 29.

In two-column addition, the procedures shown in **figure 2** have been observed. When multiplication problems such as 125 × 4 are given, children also work from left to right (see **fig. 3**).

When the problems are in division, the law of the land suddenly changes and the rule decrees that students work from left to right. If they are encouraged to do their own thinking, however, children proceed from right to left, as can be seen in the following examples with the problem 74 ÷ 5:

5 + 5 + 5 + 5 + 5 + 5 + ···
until the total comes close to 74.
(Children usually count on their fingers saying, "Five, ten, fifteen, twenty . . . ")

5 + 5 + 5 + 5 + 5 = 25
counting on five fingers.
If 5 fives is 25, 10 fives is 50.
Four more fives is 20, and 50 + 20 = 70.
So the answer is 14 fives, with a remainder of 4.

The preceding methods later become shortened to

10 × 5 = 50, 4 × 5 = 20
so the answer the answer is 14 with a remainder of 4.

FIGURE 1

Two invented procedures for solving $\begin{array}{r}53\\ -24\\ \hline\end{array}$ reported by Madell (1985)

50 − 20 = 30	50 − 20 = 30
30 − 4 = 26	4 − 3 = 1
26 + 3 = 29	30 − 1 = 29

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FIGURE 2

Three invented procedures for solving $\begin{array}{r}18\\+17\\\hline\end{array}$

10 + 10 = 20	10 + 10 = 20	10 + 10 = 20
8 + 7 = 15	8 + 2 = another ten	7 + 7 = 14
20 + 10 = 30	20 + 10 = 30	14 + 1 = 15
30 + 5 = 35	30 + 5 = 35	20 + 10 = 30
		30 + 5 = 35

FIGURE 3

Two invented procedures for solving 125 × 4

4 × 100 = 400	4 × 100 = 400
4 × 20 = 80	4 × 25 = 100
4 × 5 = 20	400 + 100 = 500
400 + 80 + 20 = 500	

Getting Children to Invent

The authors' way of teaching is not exactly the same as Madell's, for theoretical reasons. First, we do not let children write anything (until the numbers get too big to remember) because we want them to think and to talk to each other. Second, we do not use base-ten blocks because (*a*) the source of logico-mathematical knowledge is the child's mental action rather than the objects in the external world and (*b*) "one ten" is a new, higher-order construction, rather than ten *ones* merely stuck together (Kamii 1989a; Kamii and Joseph 1988).

At the beginning of second grade, the teacher writes one problem after another, such as the following, on the chalkboard, and asks, "What's a quick and easy way of solving this problem?"

$$\begin{array}{r}9\\+5\\\hline\end{array}\qquad\begin{array}{r}4\\7\\5\\2\\5\\+3\\\hline\end{array}\qquad\begin{array}{r}15\\+6\\\hline\end{array}\qquad\begin{array}{r}13\\+13\\\hline\end{array}\qquad\begin{array}{r}18\\+14\\\hline\end{array}$$

The entire class can work together, or the teacher can work with small groups. The children raise their hands when they have an answer.

When most of the hands are up, the teacher calls on individual children and writes all the answers given by them. Being careful not to say that an answer is right or wrong, the teacher then asks for an explanation of each procedure used by the children. For the first problem (9 + 5 written vertically), for example, if a child says, "I take one from the five to make ten," the teacher crosses out the 5 and the 9 and writes "10" next to the 9. If the child then says, "That makes the five be four," the teacher writes "4" below the 10. If the child concludes by saying, "Ten and four is fourteen," the teacher draws a line below the 4 and writes the answer, "14," below this line as well as below the line in the original problem.

As the teacher thus interacts with the volunteer, he or she encourages the rest of the class to express agreement or disagreement and to speak up immediately if something does not make sense. The exchange of points of view is very important in a constructivist program, and the teacher is careful not to reinforce right answers or to correct wrong ones. If the teacher were to judge correctness of answers, the children would come to depend on him or her to know whether an answer is correct. If the teacher does not say that an answer is correct or incorrect and encourages the children to agree or disagree among themselves, the class will continue to think and to debate until agreement is reached.

Many teachers ask, "What should the teacher do if no one in the class gets the right answer?" The reply is that if this happened, the teacher would know that the problem was too hard for the class and would go on to something else. In the logico-mathematical realm, if children debate long enough, they will eventually get to the correct answer because absolutely nothing is arbitrary in logico-mathematical knowledge. For example, 18 plus 14 equals 32 in every culture because nothing is arbitrary in this relationship. The reader interested in more detail about this point and this method of teaching is referred to Kamii (1989a, 1989b, 1990a, 1990b).

Advantages of Child-invented Procedures

The authors think it is better for children to invent their own procedures for three reasons. These are summarized first and elaborated on later. When children invent their own ways,

1. they do not have to give up their own thinking;
2. their understanding of place value is strengthened rather than weakened by algorithms; and
3. they develop better number sense.

It must be clear from the previous discussion that when children are encouraged to invent their own ways of solving problems, they do not have to give up their own ways of thinking. Referring to the algorithms that children are made to use, Madell said, "The early focus on memorization in the teaching of arithmetic thoroughly distorts in children's minds the fact that mathematics is primarily reasoning. This damage is often difficult, if not impossible, to undo" (1985, 22). The authors agree with Madell and add that they have learned from experience that the damage is much harder to undo (Kamii and Lewis 1993) than imagined when first reading Madell's article.

> We want children to think and to talk to each other.

The second reason it is better to encourage children to do their own thinking is that when thinking in their own ways, they strengthen their knowledge of place value by using it. When students in the constructivist program solve problems such as

$$\begin{array}{r} 987 \\ +654 \\ \hline \end{array},$$

they think and say, for example, "Nine hundred and six hundred is one thousand five hundred. Eighty and fifty is a hundred thirty; so that's one thousand six hundred thirty. Plus eleven is one thousand six hundred forty-one." By contrast, many of the children who use the algorithm unlearn place value by saying, for example, "Seven and four is eleven. Put one down and one up. One and eight and five is fourteen. Put four down and one up. One and nine is ten, so that's sixteen." Note that this algorithm is convenient for adults, who already know place value. For children, who have a tendency to think about every column as ones, the algorithm reinforces this weakness.

Let us examine the knowledge of place value among the children at Hall-Kent School. As can be seen in the following distribution for 1989–91, the constructivist teachers, who chose not to teach algorithms, tended to be in the lower grades: first grade, four out of four teachers; second grade, two out of three teachers; third grade, one out of three teachers; and fourth grade, none of the four teachers.

Children were assigned to classes as randomly as possible by the principal at the beginning of the school year. In second grade, students were taught algorithms in one of the three classes (class 1) and not in the remaining two. The remaining two classes differed slightly in that the teacher of class 2 did not call parents to discourage their use of home-taught algorithms, whereas the teacher of class 3 did.

In individual interviews in May 1990, the second graders were shown a sheet of paper on which "7 + 52 + 186" was written horizontally. They were asked to solve the problem without paper and pencil, give the answer, and then explain how they got the answer. The interviewer took notes on what each child said.

The children in class 1 used the algorithm and typically said, "Seven and two and six is fifteen. Put down the five, and carry one. One and five and eight is fourteen, put down the four. . . . This is hard. . . . I forgot what I put down before." The children in class 3, which will be called the constructivist class, typically said, "One hundred eighty and fifty is two hundred thirty. Two hundred thirty-seven, two hundred thirty-nine, two hundred forty-five."

Insight can be gained about children's understanding of place value by analyzing the wrong answers they gave. In the algorithm class, the wrong answers tended to be very small or very large. Three children got small totals of 29 or 30 by adding all the digits as ones ($7 + 5 + 2 + 1 + 8 + 6 = 29$). At the other extreme, seven children in the algorithm class gave large totals ranging from 838 to 9308. Totals in the 800s were obtained by adding the 7 and the 1 of 186. If children carried 1 from the tens column, their total came out in the 900s. By contrast, most of the wrong answers found in the constructivist class were more reasonable and ranged from 235 to 255. (The percent getting the correct answer were 12 percent in the algorithms class [class 1] and 45 percent in the constructivist class [class 3].)

> The class is encouraged to speak up immediately.

Class 2 came out in between, and the wrong answers given by this group fell between the ranges of those of classes 1 and 3. (The percent getting the correct answer was 26.)

Similar results were found by giving a similar problem (6 + 53 + 185) in May 1991 to four fourth-grade classes, all of which had been taught algorithms. The errors of the fourth-grade classes larger than the largest error of 617 produced by the second-grade constructivist class were 713 + 8, 715, 744, 814, and 1300 in one class; 713, 718, 783, 783, 783, 844, 848, and 1215 in the second class; 718, 721, 738, 738, and 791 in the third class; and 745, 835, 838, 838, and 10 099 in the fourth class. The fourth graders who were taught algorithms did considerably worse than the second graders who did their own thinking. (The percent of fourth graders who got the correct answer of 244 were only 24, 17, 30, and 19, respectively, in the four classes.)

It is clear from examining the answers given to the preceding problems that children who know place value also have better number sense. Because those who do their own thinking usually start with larger units, such as 180 + 50, they are not likely to get answers in the 700s, 800s, or beyond (for 6 + 53 + 185). When so many fourth graders get answers in the 700s and 800s, it seems apparent that algorithms unteach place value and prevent children from developing number sense.

The better number sense of children who do their own thinking also comes from the fact that they think about entire numbers and not about each column separately. Responses to the following problem illustrate this point:

$$\begin{array}{r} 504 \\ -306 \\ \hline \end{array}$$

Most of the second and third graders (74 percent and 80 percent, respectively) who had never been taught algorithms easily got the correct answer by doing $500 - 300 = 200$, $4 - 6 = -2$, $200 - 2 = 198$. The fourth graders, who used the algorithm, again did much worse. The percent of correct responses was 29, 38, 39, and 55, respectively, for the four fourth-grade classes.

The children's wrong answers revealed their number sense. The greatest wrong answer found among the constructivist third graders was 202. By contrast, the fourth graders, who used the algorithm, got larger wrong answers, such as 208 (10 percent of all the answers), 298 (6 percent of all the answers), 308, 408, 410, 498, 808, and 898. Whereas the smallest wrong answer found among the constructivist third graders was 190, the fourth graders, who used algorithms, got smaller wrong answers, such as 108 (15 percent of all the answers), 148, and 189 (4 percent of all the answers). Because they thought only of isolated columns, they did not sense anything wrong even when they were unreasonably off the mark.

When third graders were given the multiplication problem 13×11, 60 percent of those who had never been taught algorithms got the correct answer by thinking $13 \times 10 = 130$, $130 + 13 = 143$. Although almost all the fourth graders could get the correct answer by using the algorithm, only the following percent of the four classes got the correct answer when they were allowed to use only their heads: 5, 6, 14, and 15. The incorrect answers given by the fourth graders again demonstrated their lack of number sense. The incorrect answers were 11, 13, 23, 26, 33, 42, 44, 45, 64, 66, 113, 123, 131, 133, 140, 141, 155, 1300, and 1313.

The view that children should be encouraged to do their own thinking is now advocated by many other educators and researchers working from a variety of theoretical perspectives. This view is supported not only in the United States (Cobb and Wheatley 1988; Lester 1989) but also in Brazil (Carraher, Carraher, and Schliemann 1985, 1987; Carraher and Schliemann, 1985), England (Plunkett 1979), Holland (Gravemeijer 1990; Heege 1978; Streefland 1990; Treffers 1987), Mexico (Ferreiro 1988), and South Africa (Murray and Olivier 1989; Olivier, Murray, and Human 1990, 1991). If we are serious about reform in mathematics education, we must study how young children think and reexamine our fundamental beliefs about teaching.

> Children develop good number sense when devising their own procedures.

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